DSC 40A

Theoretical Foundations of Data Science I

Foundations of Probability – Conditional Probability

Announcements

- Homework 5 released today
- Homework 4 grades will be released today.
- No in-person discussion section on 11/11 - the groupwork will instead be asynchronous and due on Wed. 11/13 at 11:59PM.

Agenda

- Multiplication rules and independence
- Conditional probability

Question

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "Lecture Questions" link in the top right corner of dsc40a.com.

Multiplication rule and Independence

- The probability that events A and B both happen is

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$$

AND

- $\mathbb{P}(B|A)$ means "the probability that B happens, given that A happened." It is a **conditional probability**.
    - More on this soon!
- If $\mathbb{P}(B|A) = \mathbb{P}(B)$, we say A and B are **independent**.
    - Intuitively, A and B are independent if knowing that A happened gives you no additional information about event B, and vice versa.
    - For two independent events, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Practice Problems

Example 2. A die is rolled 3 times. What is the probability that the face 1 never appears in any of the rolls?

$S = \{\text{all trios of rolls}\}$

$E = \{\text{not 1 in roll 1 and not 1 in roll 2 and not 1 in roll 3}\} =$

$P(\text{not 1 in roll 1}) \cdot P(\text{not 1 in roll 2}) \cdot P(\text{not 1 in roll 3}) = \left(\frac{5}{6}\right)^3$

die rolls are independent

$P(A \cap B \cap C) = P(A)P(B)P(C)$

$P(\text{not } 1) = 1 - P(\text{rolled } 1) = 1 - \frac{1}{6} = \frac{5}{6}$

$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \text{ and } B) =$

$P(A) \cdot P(B) \cdot P(C)$

Practice Problems

Example 3. A die is rolled n times. What is the chance that only faces 2, 4 or 6 appear?

S for a single roll $= \{1, 2, 3, 4, 5, 6\}$

$E = \{2, 4, 6\}$ for a single roll

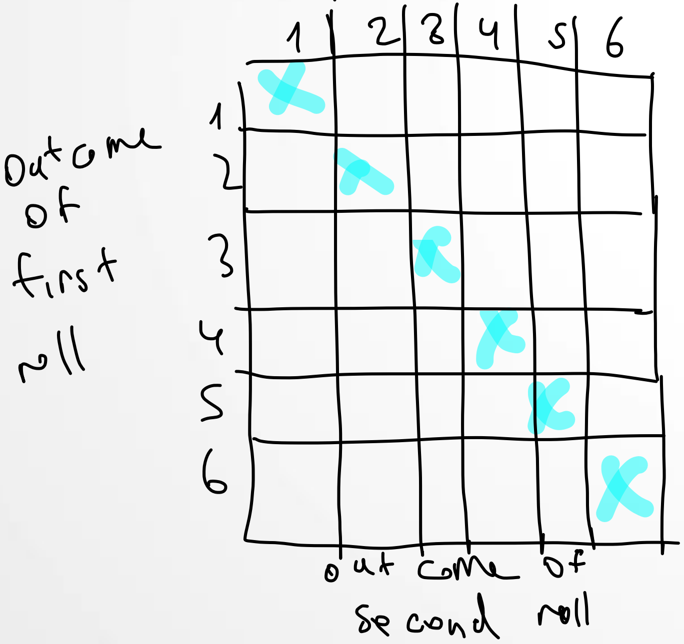
$$P(E) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} = \frac{\#\text{outcomes in } E}{\#\text{outcomes in } S}$$

$$\underbrace{P(E) \cdot P(E) \cdot P(E) \cdot \ldots}_{n \text{ times}} \cdot P(E) = \left(\frac{1}{2}\right)^n$$

Practice Problems

Example 4. A die is rolled two times. What is the probability that the two rolls had different faces?

S = all pairs of rolls



$E = \{$all pairs with different faces$\}$

$\bar{E} = \{$both rolls had same face$\}$

$= \{11, 22, 33, 44, 55, 66\}$

$p(\bar{E}) = \frac{6}{36} = \frac{1}{6}$

$p(E) = 1 - p(\bar{E}) = 1 - \frac{1}{6} = \frac{5}{6}$

S = 2nd roll

$\{1, 2, 3, 4, 5, 6\}$

E = 2nd roll different from first

$p(E) = \frac{5}{6}$

from Theory Meets Data by Ani Adhikari, Chapter 4

Conditional probabilities

Probability of an event may **change** if have additional information about outcomes.

Rolling a die

What is the probability of rolling a number greater than 3?

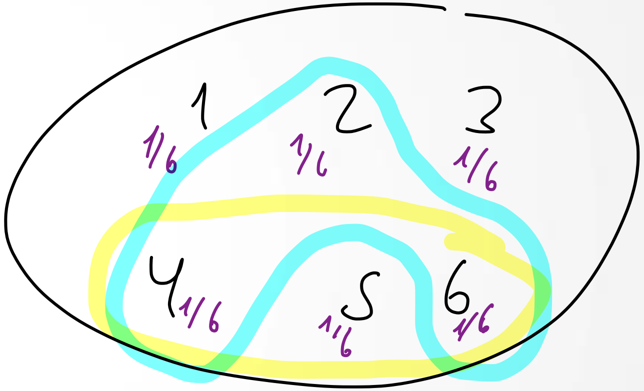
$S = \{1, 2, 3, 4, 5, 6\}$

$E = \{4, 5, 6\}$

$$p(E) = \sum_{s \in E} p(s) = \frac{3}{6} = \frac{1}{2}$$

$$= \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S}$$

$S =$



Extra information: outcome was even

$F = \{2, 4, 6\}$

$P(E \mid F) = ?$

Conditional probabilities

Probability of an event may **change** if have additional information about outcomes.

Suppose E and F are events, and P(F) > 0. Then,

$E = \{4, 5, 6\}$

$F = \{2, 4, 6\}$

given that

intersection

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

AND

$P(F) = \frac{3}{6} = \frac{1}{2}$

$P(E \cap F) = \frac{2}{6}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{6}}{\frac{1}{2}} = \frac{4}{6} = \frac{2}{3} > \frac{1}{2} = P(E)$$

i.e.,

$$P(E \cap F) = P(E|F)P(F)$$

$$= P(F|E) \cdot P(E) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

Conditional probabilities

Are these probabilities equal?

$p(c) = p(d) = \frac{1}{2}$

Suppose a family has two pets. Assume that it is **equally likely** that each pet is a dog or a cat. Consider the following two probabilities:

- The probability that both pets are dogs given that **the oldest is a dog.**
- The probability that both pets are dogs given that **at least one of them is a dog.**

What do you think?

A. they are equal
B. they are not equal

Conditional probabilities

Are these probabilities equal?

Suppose a family has two pets. Assume that it is **equally likely** that each pet is a dog or a cat. Consider the following two probabilities:

- The probability that both pets are dogs given that **the oldest is a dog.**
- The probability that both pets are dogs given that **at least one of them is a dog.**

$S = \{\underset{1/4}{dd}, \underset{1/4}{cd}, \underset{1/4}{dc}, \underset{1/4}{cc}\}$

$E = \{dd\}$

$F = \{dd, dc\}$

$E \subset F$

$P(E) = \frac{1}{4}$

$P(F) = \frac{2}{4} = \frac{1}{2}$

$P(E \mid F) = \frac{P(E \cap F)}{P(F)} =$

$= \frac{P(E)}{P(F)} = \frac{1/4}{1/2} = \frac{1}{2}$

$E \cap F = E$

Conditional probabilities

Are these probabilities equal?

Suppose a family has two pets. Assume that it is **equally likely** that each pet is a dog or a cat. Consider the following two probabilities:

- The probability that both pets are dogs given that **the oldest is a dog.**
- The probability that both pets are dogs given that **at least one of them is a dog**.

$S = \{dd, cd, dc, cc\}$

$E = \{dd\}$ $\quad P(E) = \frac{1}{4}$

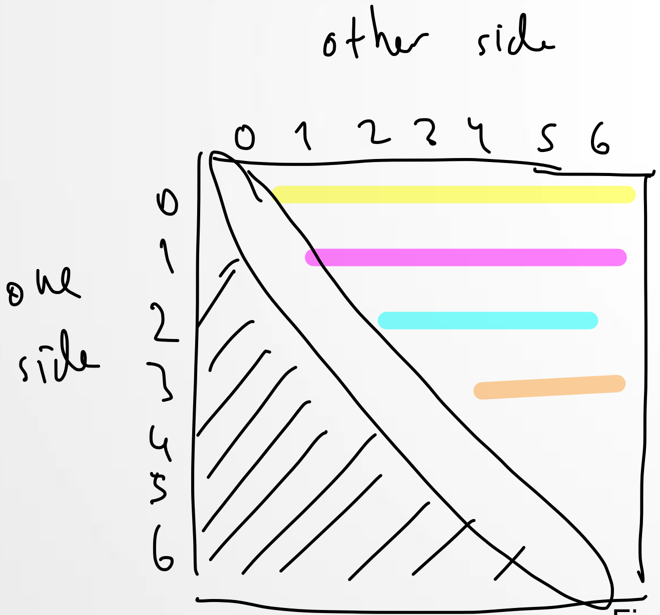
$F = \{dd, dc, cd\}$ $\quad P(F) = \frac{3}{4}$

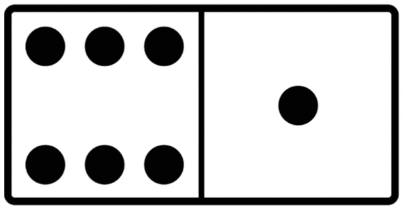
$E \cap F = E$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Dominoes

In a set of dominos, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.





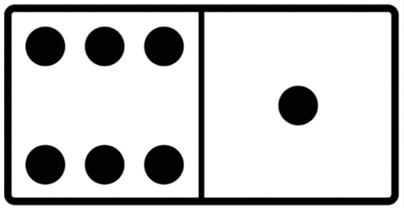
Why 28?

$6+5+4+3+2+1+7=28$ tiles

Five Thirty Eight Riddler Express: https://fivethirtyeight.com/features/can-you-eat-an-apple-like-a-toddler/

Dominoes

In a set of dominos, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.



Question 1: What is the probability of drawing a “double” from a set of dominoes — that is, a tile with the same number on both sides?

Question 2: Now you pick a random tile from the set and uncover only one side, revealing that it has six dots. What’s the probability that this tile is a double, with six on both sides?

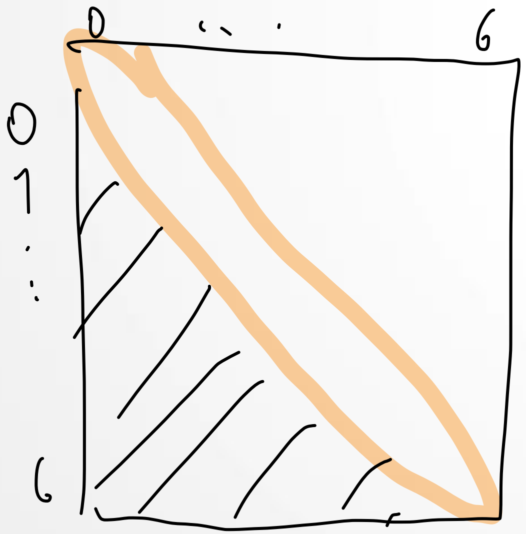
Question 3: Now your friend picks a random tile from the set and looks at it. You ask if they have a six, and they answer yes. What is the probability that your friend’s tile is a double, with six on both sides?

Five Thirty Eight Riddler Express: https://fivethirtyeight.com/features/can-you-eat-an-apple-like-a-toddler/

Dominoes

Question 1: What is the probability of drawing a “double” from a set of dominoes — that is, a tile with the same number on both sides?

$S = \{\text{all } 28 \text{ tiles}\}$



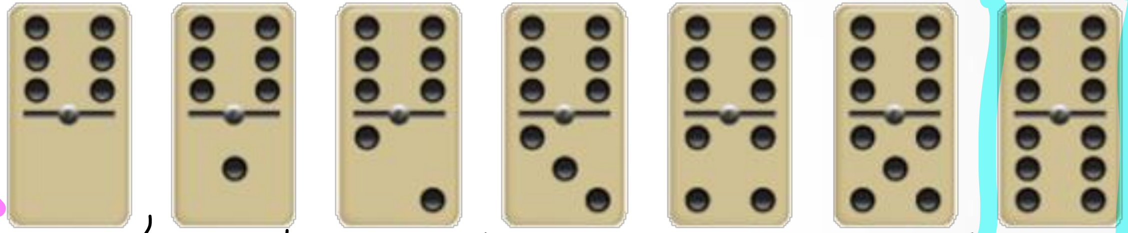
$E = \{00, 11, 22, 33, 44, 55, 66,\}$

$$P(E) = \frac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in } S} = \frac{7}{28} = \frac{1}{4}$$

Five Thirty Eight Riddler Express: https://fivethirtyeight.com/features/can-you-eat-an-apple-like-a-toddler/

Dominoes

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?

$F = \{$  $\}$

$S = \{28 \text{ tiles}\}$

$E = \{\text{double tiles} = 66\}$

$F = \{\text{domino with at least one six}\}$

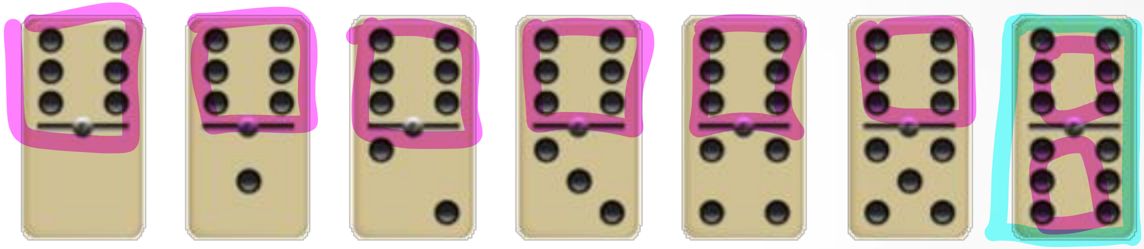
$$P(F) = \frac{\#\text{outcomes in } F}{\#\text{outcomes in } S} = \frac{7}{28} = \frac{1}{4}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/28}{7/28} = \frac{1}{7}$$

$$P(E \cap F) = P(E) = \frac{1}{28}$$

Dominoes

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has 6 dots. What is the probability that this tile is a double, with 6 on both sides?



$p = \frac{1}{4}$ why?

Five Thirty Eight Riddler Express: https://fivethirtyeight.com/features/can-you-eat-an-apple-like-a-toddler/

Conditional probabilities: Simpson's Paradox

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Which treatment is better?

A. Treatment A for all cases.
B. Treatment B for all cases.
C. A for small and B for large.
D. A for large and B for small.

C. R. Charig, D. R. Webb, S. R. Payne, J. E. Wickham (29 March 1986). "Comparison of treatment of renal calculi by open surgery, percutaneous nephrolithotomy, and extracorporeal shockwave lithotripsy". Br Med J (Clin Res Ed) 292 (6524): 879–882. doi:10.1136/bmj.292.6524.879. PMC 1339981. PMID 3083922. cf. Wikipedia "Simpson's Paradox"

Conditional probabilities: Simpson's Paradox

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Simpson's Paradox

"When the less effective treatment is applied more frequently to easier cases, it can appear to be a more effective treatment."

C. R. Charig, D. R. Webb, S. R. Payne, J. E. Wickham (29 March 1986). "Comparison of treatment of renal calculi by open surgery, percutaneous nephrolithotomy, and extracorporeal shockwave lithotripsy". Br Med J (Clin Res Ed) 292 (6524): 879–882. doi:10.1136/bmj.292.6524.879. PMC 1339981. PMID 3083922. cf. Wikipedia "Simpson's Paradox"

Summary

- Today, we studied conditional probability.

- **Next time:** How do we use probability to answer questions about random samples?