DSC 40A - Probability Roadmap

A hopefully helpful guide to solving probability questions

1. What is the sample space, S (all possible outcomes)? Consider using:

- individual objects
- sequences of objects
- sets of objects
- sequences of positions
- sets of positions

Example: Suppose that a standard deck of 52 cards is shuffled in a random order. What is the probability that both red queens are adjacent?

Try to do this problem as many ways as you can, using a different sample space each time.

2. Are all of the outcomes in S equally likely, or are some more likely than others?

If all equally likely (most common), can use

$$P(\text{favorable outcome}) = \frac{\text{number of favorable outcomes in } S}{\text{total number of outcomes in } S}.$$

Then our probability question turns into two counting questions.

If not all equally likely, add up probabilities associated with each of the favorable outcomes, using

$$P(E) = \sum_{s \text{ in } E} P(s).$$

Example: Consider the sample space $S = \{t, u, v, w, x, y, z\}$ with associated probabilities given in the table below.

outcome	t	u	v	w	x	y	z
probability	$\frac{5}{21}$	$\frac{2}{21}$	$\frac{1}{21}$	$\frac{4}{21}$	$\frac{2}{21}$	$\frac{4}{21}$	$\frac{3}{21}$

Let $A = \{t, u, v, w\}$ and $B = \{v, w, x, y\}$. Find $P(A|B)$.

3. What are the favorable outcomes in S? Try to write down one favorable outcome, then think about what was forced and what you were free to select in generating this example. This can help you count the number of favorable outcomes.

Example: If favorable outcomes are sequences of 52 cards with both red queens adjacent, list one favorable outcome and count the number of favorable outcomes.

Example: If favorable outcomes are hands of 7 cards with exactly 3 red cards, list one favorable outcome and count the number of favorable outcomes.

Example: If favorable outcomes are six-letter strings with two letters alternating, list one favorable outcome and count the number of favorable outcomes.

4. Is it helpful to use the complement rule? Are there many ways in which the favorable outcome can be achieved, and only a few ways in which it's not achieved? If so, try using

$$P(A) = 1 - P(\overline{A}).$$

The words "at least" often indicate to use the complement rule.

Example: If a license place is made up of three letters followed by three numbers, like ABC-123, what is the probability of a randomly selected license plate having a palindrome in the letters or numbers?

Example: What is the probability of at least one heads in k tosses of a fair coin?

Example: Every time you call your grandma, she has a $\frac{1}{3}$ chance of answering the phone. If you call your grandma three times this week, what's the probability you'll get to talk to her this week?

5. Is it helpful to use the multiplication rule? Do you need several things to happen all at once?

The multiplication rule says you can make the first thing happen, then you only have to worry about the second thing happening in the case that the first already happened.

$$P(A \cap B) = P(A) * P(B|A)$$

We can also use it in the other order:

$$P(A \cap B) = P(B) * P(A|B)$$

We can also use it for three or more events, by just making sure that the next thing happens, assuming that all prior things already happened. For example, for three events, the multiplication rule becomes

$$P(A \cap B \cap C) = P(A) * P(B|A) * P(C|(A \cap B)).$$

Example: Suppose you draw a random sample of 5 objects with replacement from a set of 20 distinct objects. What is the probability of having no repeated objects in the sample?

6. Do you need to calculate a conditional probability, the probability of an event happening when you know another has occurred? The words “given that”, “assuming”, and “if you know” are often hints that you’re being asked to find a conditional probability.

The formula for conditional probability is just the same as the multiplication rule, written in a slightly different way:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

When computing a conditional probability, you can either:

1. use the above formula, or
2. calculate the conditional probability directly.

For most problems, calculating the conditional probability directly is usually easier because using the formula requires you to find two probabilities, one for the numerator and one for the denominator, while a direct calculation is just one probability. Also, finding the probability of the intersection of two events can be harder than just finding the conditional probability.

When calculating a conditional probability directly, use the given information (the condition) to restrict the possible set of outcomes.

Example: A license place is made up of three letters followed by three numbers, like ABC-123. If you know that a randomly selected license plate starts with an A, what is the probability of that license plate having three distinct letters that appear in alphabetical order?

Try this problem both ways, using the conditional probability formula and calculating the conditional probability directly. Which way do you think is easier?

7. Is it helpful to use the addition rule? Can you break down the favorable outcomes into a reasonable number of cases? Do the cases overlap? Can any one outcome fall into multiple cases? If so, don’t forget to subtract off the intersection to avoid counting these outcomes multiple times.

The basic addition rule for two cases is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If the cases don’t overlap, the last term is zero and can be left out.

For more than two cases, if the cases don’t overlap, you can simply add up the individual properties of each case. If they do overlap, the formula gets complicated very quickly. You only need to know up to three cases:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(C \cap B) + P(A \cap B \cap C).$$

Example: What is the probability that a random five-letter string ends in a or z?

Example: What is the probability that a random five-letter string starts in a or ends in z?

Example: Every time you call your grandma, she has a $\frac{1}{3}$ chance of answering the phone. If you call your grandma three times this week, what's the probability you'll get to talk to her this week?

8. If you're counting sequences, can the sequences be organized according to templates? Templates are like cases, showing a pattern that the sequence will follow. Templates should be disjoint, not overlapping.

If each template is associated with the same probability, then each term in the addition rule has the same value so we can calculate the probability as:

$$(\text{number of templates}) * (\text{probability of each template}).$$

Example: Suppose that every time you visit a volcano, you have a one percent chance of witnessing an eruption. If you visit five times, what is the probability of seeing your second eruption on your fifth visit?

Example: All UCSD campus phone numbers take the form 858-534-XXXX. What is the probability of a randomly chosen UCSD phone number having each digit that's included in the ten-digit number appearing exactly twice?

9. Can you think of an additional assumption that would make the probability easier to calculate? Can you calculate how likely that assumption is to be true? If so, you may be able to break up your probability into cases based on this extra assumption and whether it is true or not.

This approach basically uses the Law of Total Probability. If A is the event you want the probability of, and B is the extra assumption, then since B and $\overline{B}$ partition the sample space, the Law of Total Probability says

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap \overline{B}) \\ &= P(A|B) * P(B) + P(A|\overline{B}) * P(\overline{B}) \end{aligned}$$

Example: Suppose that a standard deck of 52 cards is shuffled in a random order. What is the probability that both red queens are adjacent?

Example: Every time you call your grandma, she has a $\frac{1}{3}$ chance of answering the phone. If you call your grandma three times this week, what's the probability you'll get to talk to her this week?