# DSC 40A - Group Work Session 7

(not due)

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **tag all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive full credit, you must work in a group of two to four students for at least 50 minutes in your assigned discussion section. You can also self-organize a group and meet outside of discussion section for 80 percent credit. You may not do the groupwork alone.

## 1 Combinatorics

In probability, when all outcomes in the sample space are equally likely, the probability of an event is the number of outcomes in the event divided by the number of outcomes in the sample space. Thus, the probability of the event reduces to two counting (or combinatorics) questions, which ask *how many* outcomes are possible. When solving a counting question, it helps to write down one example outcome, then try to think about how many options we had at each step of generating this example.

There are a few basic combinatorial objects that we've studied in this class, namely sequences, permutations, and combinations. The hard part is often determining which one to use in which situation, which comes down to two important questions:

- Does the order in which I select the objects matter? In other words, does it count as different or the same if I choose the same objects in a different order?
- Am I selecting objects with or without replacement? In other words, am I allowed to have repeated objects?

**Problem 1. Xs and Os**

Let $N(a, b)$ represent the number of strings you can create out of $a$ Xs and $b$ Os. Explain why $N(a, b)$ satisfies each of the following:

$$N(0, b) = 1 \tag{1}$$
$$N(a, 0) = 1 \tag{2}$$
$$N(a, b) = N(a-1, b) + N(a, b-1) \quad \text{for } a > 0 \text{ and } b > 0. \tag{3}$$

**Problem 2. Tiebreaker**

To break a tie among a group of $n \geq 3$ people, you come up with the following tiebreaker: Everyone flips a coin. If one person's coin is different from all the others, that person wins, and the tie is broken! Otherwise, repeat the process.

**a)** What is the probability that the tie is broken after the first coin toss?

**b)** Fix an integer $k \geq 1$. Find the probability that the tie is broken after exactly $k$ coin tosses?

## 2 Bayes' Theorem and the Law of Total Probability

**Bayes' Theorem** describes how to update the probability of one event given that another has occurred.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B) \times \mathbb{P}(B)}{\mathbb{P}(A)}$$

You can think of Bayes' Theorem as a restatement of the multiplication rule. If we multiply both sides of the expression above by $\mathbb{P}(A)$, we get two equivalent expressions for $\mathbb{P}(A \cap B)$ using the multiplication rule.

Another useful rule that is helpful in many Bayes' Theorem problems is the **Law of Total Probability** which says that if we have events $E_1, E_2, \ldots, E_k$ that partition our sample space,

$$\mathbb{P}(A) = \mathbb{P}(A \cap E_1) + \mathbb{P}(A \cap E_2) + \cdots + \mathbb{P}(A \cap E_k).$$

**Problem 3.**

There are two boxes. Box 1 contains three red and five white balls and box 2 contains two red and five white balls. A box is chosen at random, with each box equally likely to be chosen. Then, a ball is chosen at random from this box, with each ball equally likely to be chosen. The ball turns out to be red. What is the probability that it came from box 1?

## 3 Independence and Conditional Independence

Recall that two events $A$, $B$ are **independent** if knowledge of one event occurring does not affect the probability of the other event occurring. There are three equivalent definitions of independence:

$$\mathbb{P}(A|B) = \mathbb{P}(A) \tag{4}$$
$$\mathbb{P}(B|A) = \mathbb{P}(B) \tag{5}$$
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B) \tag{6}$$

Two events that are not independent are also called **dependent**.

Two events $A$ and $B$ are **conditionally independent given** $C$ if

$$\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \times \mathbb{P}(B|C).$$

Notice the similarity between this definition and the third definition of independence given above. Conditional independence given $C$ means that *when $C$ occurs*, $A$ and $B$ are independent in that case. But they may or may not be independent in general.

**Problem 4.**

Let $A$ and $B$ be events in a sample space with $0 < \mathbb{P}(A) < 1$ and $0 < \mathbb{P}(B) < 1$. If $A$ is a subset of $B$, can $A$ and $B$ be independent? If yes, give an example, otherwise prove why not.

**Problem 5.**

Consider two flips of a fair coin. The sample space is $S =$ all outcomes of 2 flips of a coin $= \{HH, HT, TH, TT\}$, where each has equal probability $\frac{1}{4}$. We define the event $A$ as $A =$ first flip is heads $= \{HH, HT\}$, and the event $B$ as $B =$ second flip is heads $= \{HH, TH\}$. You can verify that $A$ and $B$ are independent by showing $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$.

Now, suppose that the coin is not fair, and instead flips heads the first time with probability $p$ and flips heads the second time with probability $q$.

Are $A$ and $B$ still independent?

**Problem 6.**

A box contains two coins: a regular coin and one fake two-headed coin ($\mathbb{P}(H) = 1$). Choose a coin at random and flip it twice. Define the following events.

- A: First flip is heads (H).
- B: Second flip is heads (H).
- C: Coin 1 (regular) has been selected.

Are $A$ and $B$ independent? Are $A$ and $B$ conditionally independent given $C$?

Prove your answers using the definitions of independence and conditional independence. Also explain your answers intuitively.