Lecture 10

Feature Engineering, Gradient Descent

DSC 40A, Spring 2024

Announcements

- Homework 4 is due **tonight**.
  - Some office hours are now in HDSI **355** – see the calendar for more details.
- Homework 2 scores are available on Gradescope.
  - Regrade requests are due on Monday.
- We will have a review session on **tomorrow from 2-5PM in Center Hall 109** where we'll go over old homework and exam problems.
  - It'll be recorded!

The Midterm Exam is on Tuesday, May 7th!

- The Midterm Exam is on **Tuesday, May 7th in class**.
    - You must take it during your scheduled lecture session.
    - You will receive a randomized seat assignment over the weekend.
- 80 minutes, on paper, no calculators or electronics.
    - **You are allowed to bring one two-sided index card (4 inches by 6 inches) of notes that you write by hand (no iPad).**
- Content: Lectures 1-9, Homeworks 1-4, Groupworks 1-4.
- Prepare by practicing with old exam problems at practice.dsc40a.com.
    - Problems are sorted by topic!

Agenda

- Feature engineering and transformations. → in scope!
- Minimizing functions using gradient descent. → not in scope for the midterm!

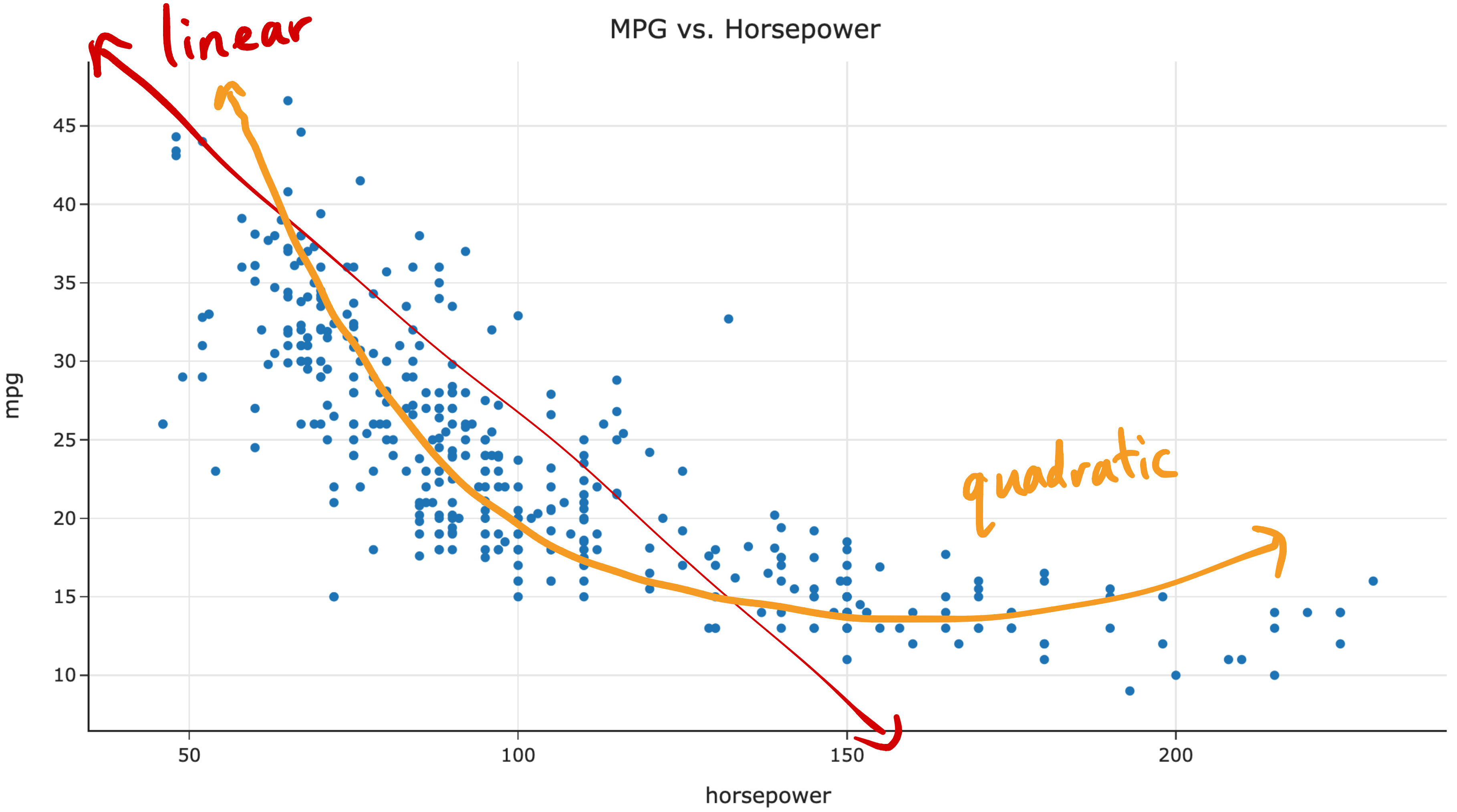
Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

Feature engineering and transformations



Question: Would a linear hypothesis function work well on this dataset?

Need $\vec{h} = X\vec{w}$

→ preds = dot product of row of X with $H(\vec{x}) = \text{Aug}(\vec{x}) \cdot \vec{w}$

$w_0 + w_1 \cdot \square + w_2 \cdot \square + w_3 \cdot \square + \ldots$

$\square$ shouldn't involve w!

Linear in the parameters

- We can fit rules like:

$$w_0 + w_1 x + w_2 x^2 \qquad w_1 \underbrace{e^{-{x^{(1)}}^2}}_{\text{feature 1}} + w_2 \underbrace{\cos(x^{(2)} + \pi)}_{\text{feature 2}} + w_3 \underbrace{\frac{\log 2x^{(3)}}{x^{(2)}}}_{\text{feature 3}}$$

  - This includes arbitrary polynomials.
  - These are all linear combinations of (just) features.

- We can't fit rules like:

not good!

not good! can't write as $\text{Aug}(\vec{x}) \cdot \vec{w}$

$$w_0 + e^{w_1 x} \qquad w_0 + \sin(w_1 x^{(1)} + w_2 x^{(2)})$$

  - These are **not** linear combinations of just features!

- We can have any number of parameters, as long as our hypothesis function is **linear in the parameters**, or linear when we think of it as a function of the parameters.

Example: Amdahl's Law

- Amdahl's Law relates the runtime of a program on p processors to the time to do the sequential and nonsequential parts on one processor.

stuff that can be parallelized

$$H(p) = t_{\mathrm{S}} + \frac{t_{\mathrm{NS}}}{p}$$

stuff that can't be parallelized

- Collect data by timing a program with varying numbers of processors:

Processors	Time (Hours)
1	8
2	4
4	3

linear in the parameters!

Example: Fitting $H(x) = w_0 + w_1 \cdot \frac{1}{x}$

Processors	Time (Hours)
1	8
2	4
4	3

x $\qquad$ y

$$X = \begin{bmatrix} 1 & 1/1 \\ 1 & 1/2 \\ 1 & 1/4 \end{bmatrix}_{3 \times 2} \qquad \vec{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}_{2 \times 1}$$

$$\vec{h} = X\vec{w}$$

$$\vec{y} = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}_{3 \times 1}$$

What are w_0^* and w_1^*? Find by solving

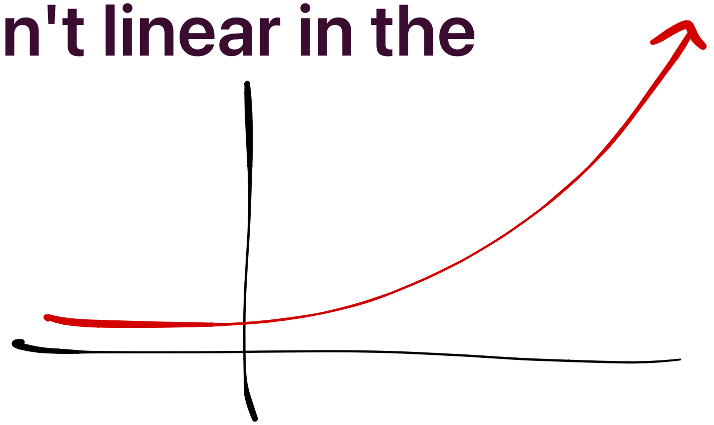
$$X^TX\vec{w}^* = X^T\vec{y}$$

system of 2 equations, 2 unknowns

$$\vec{w}^* = (X^TX)^{-1}X^T\vec{y}$$

X^TX invertible because X is full rank ⇒ all of X's columns are linearly independent!

How do we fit hypothesis functions that aren't linear in the parameters?



- Suppose we want to fit the hypothesis function:

$$H(x) = w_0 e^{w_1 x}$$

- This is **not** linear in terms of w_0 and w_1, so our results for linear regression don't apply.
- **Possible solution**: Try to apply a **transformation**.

Goal: $w_0 + w_1 \cdot \square$

Transformations

- **Question**: Can we re-write $H(x) = w_0 e^{w_1 x}$ as a hypothesis function that **is** linear in the parameters?

$$y = w_0 e^{w_1 x}$$

Try to $\log_e$ both sides!

$$\log y = \log(w_0 e^{w_1 x})$$

$$\log y = \log(w_0) + \log(e^{w_1 x})$$

$$\underbrace{\log y}_{z} = \underbrace{\log(w_0)}_{b_0} + \underbrace{w_1}_{b_1} x$$

$$z = b_0 + b_1 x$$

linear in the parameters!!!

① $\log(ab) = \log(a) + \log(b)$

$z = \log y$

$b_0 = \log w_0 \Rightarrow w_0^* = e^{b_0^*}$

$b_1 = w_1$

Transformations

- **Solution**: Create a new hypothesis function, $T(x)$, with parameters b_0 and b_1, where $T(x) = b_0 + b_1 x$.
- This hypothesis function is related to $H(x)$ by the relationship $T(x) = \log H(x)$.
- $\vec{b}$ is related to $\vec{w}$ by $b_0 = \log w_0$ and $b_1 = w_1$.
- Our new observation vector, $\vec{z}$, is $\begin{bmatrix} \log y_1 \\ \log y_2 \\ \dots \\ \log y_n \end{bmatrix}$.

Solve:

$$\boxed{X^T X \underset{\text{new parameter vector}}{\vec{b}^*} = X^T \overset{\text{new observation vector}}{\vec{z}}}$$

- $T(x) = b_0 + b_1 x$ is linear in its parameters, b_0 and b_1.
- Use the solution to the normal equations to find $\vec{b}^*$, and the relationship between $\vec{b}$ and $\vec{w}$ to find $\vec{w}^*$.

Once again, let's try it out! Follow along in this notebook.

Non-linear hypothesis functions in general

- Sometimes, it's just not possible to transform a hypothesis function to be linear in terms of some parameters.
- In those cases, you'd have to resort to other methods of finding the optimal parameters.
    - For example, $H(x) = w_0 \sin(w_1 x)$ **can't** be transformed to be linear.
    - But, there are other methods of minimizing mean squared error:

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} (y_i - w_0 \sin(w_1 x))^2$$

    - One method: **gradient descent**, the topic we're going to look at next!
- Hypothesis functions that are linear in the parameters are much easier to work with.

assume no transformations

Question 🤔

Answer at q.dsc40a.com

Which hypothesis function is **not** linear in the parameters?

- ✓ A. $H(\vec{x}) = w_1(x^{(1)}x^{(2)}) + \frac{w_2}{x^{(1)}}\sin\left(x^{(2)}\right)$
- [✓ B. $H(\vec{x}) = 2^{w_1}x^{(1)}$]
- ✓ C. $H(\vec{x}) = \vec{w} \cdot \text{Aug}(\vec{x})$
- ✓ D. $H(\vec{x}) = w_1\cos(x^{(1)}) + w_2 2^{x^{(2)}}\log x^{(3)}$
- E. More than one of the above.

$w_2 \cdot \frac{\sin(x^{(2)})}{x^{(1)}}$

Goal:

$\sum w \cdot \square$

no ws!

not already linear in the parameters, **but** you could transform it.

$= w_0 + w_1 x^{(1)} + w_2 x^{(2)} + \dots + w_d x^{(d)}$

Roadmap

- This is the end of the content that's in scope for the Midterm Exam.
- Now, we'll introduce **gradient descent**, a technique for minimizing functions that can't be minimized directly using calculus or linear algebra.
- After the Midterm Exam, we'll:
  - Finish gradient descent.
  - Look at a technique for identifying patterns in data when there is no "right answer" $\vec{y}$, called **clustering**.
  - Switch gears to **probability**.

↗ figuring out the best way to make predictions!

The modeling recipe

1. Choose a model.

① $H(x) = h$ constant

② $H(x) = w_0 + w_1 x$

simple linear regression

③ $H(\vec{x}) = w_0 + w_1 x^{(1)} + w_2 x^{(2)} + \dots + w_d x^{(d)}$

$= \vec{w} \cdot \underset{\text{augmented}}{\text{Aug}(\vec{x})}$ → prediction for a single data point

$\vec{h} = X\vec{w}$ → predictions for all n data points

2. Choose a loss function.

Ⓐ squared loss: $(\underset{\substack{\uparrow \\ \text{actual}}}{y_i} - \underset{\substack{\uparrow \\ \text{predicted}}}{H(x_i)})^2$

Ⓑ absolute loss: $|y_i - H(x_i)|$

Ⓒ 0-1 loss

Ⓓ relative squared loss: Homework 2

$\frac{(y_i - H(x_i))^2}{y_i}$

3. Minimize $\underbrace{\text{average loss}}_{\text{empirical risk}}$ to find optimal model parameters.

↳ best $w_0^*, w_1^*, \dots$ or h^*

①Ⓐ $R_{sq}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2 \Rightarrow h^* = \text{Mean}(y_1, y_2, \dots, y_n)$

②Ⓑ programming Q in HW 3

③Ⓐ $R_{sq}(\vec{w}) = \frac{1}{n} \|\vec{y} - X\vec{w}\|^2$

not on the midterm!

Minimizing functions using gradient descent

Minimizing empirical risk

- Repeatedly, we've been tasked with **minimizing** the value of empirical risk functions.
    - Why? To help us find the **best** model parameters, h^* or w^*, which help us make the **best** predictions!
- We've minimized empirical risk functions in various ways.
    - $R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$ → calculus
    - $R_{\text{abs}}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n}|y_i - (w_0 + w_1 x)|$ → for-loop, brute-forced all possible lines
    - $R_{\text{sq}}(\vec{w}) = \frac{1}{n}\|\vec{y} - X\vec{w}\|^2$ → linear algebra: spans, projections

Minimizing arbitrary functions

derivative exists, and exists everywhere

- Assume $f(t)$ is some **differentiable** single-variable function.
- When tasked with minimizing $f(t)$, our general strategy has been to:
    i. Find $\frac{df}{dt}(t)$, the derivative of f.
    ii. Find the input t^* such that $\frac{df}{dt}(t^*) = 0$.
- However, there are cases where we can find $\frac{df}{dt}(t)$, but **it is either difficult or impossible to solve** $\frac{df}{dt}(t^*) = 0$.

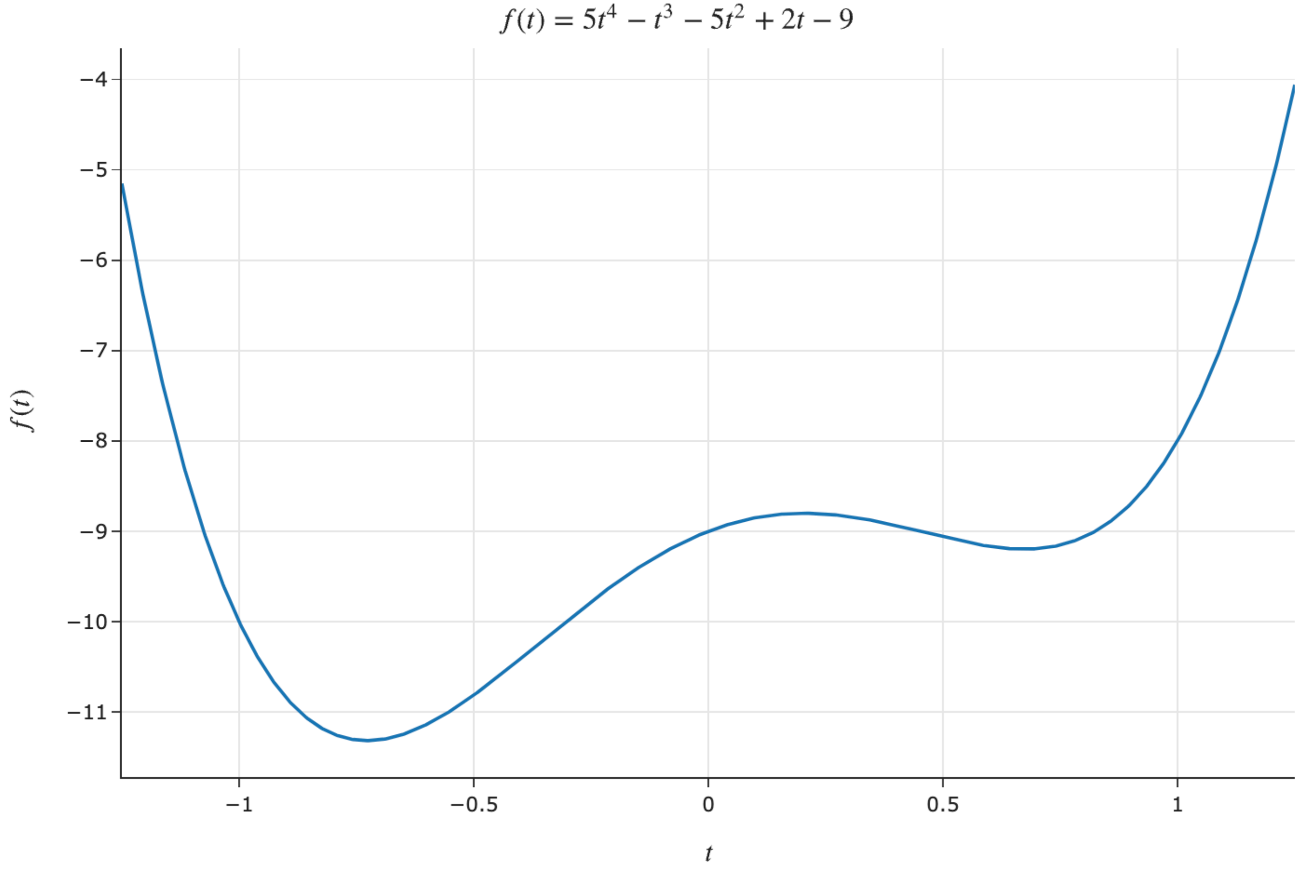
$$f(t) = 5t^4 - t^3 - 5t^2 + 2t - 9$$

$$\frac{df}{dt}(t) = 20t^3 - 3t^2 - 10t + 2$$

- Then what?

What does the derivative of a function tell us?

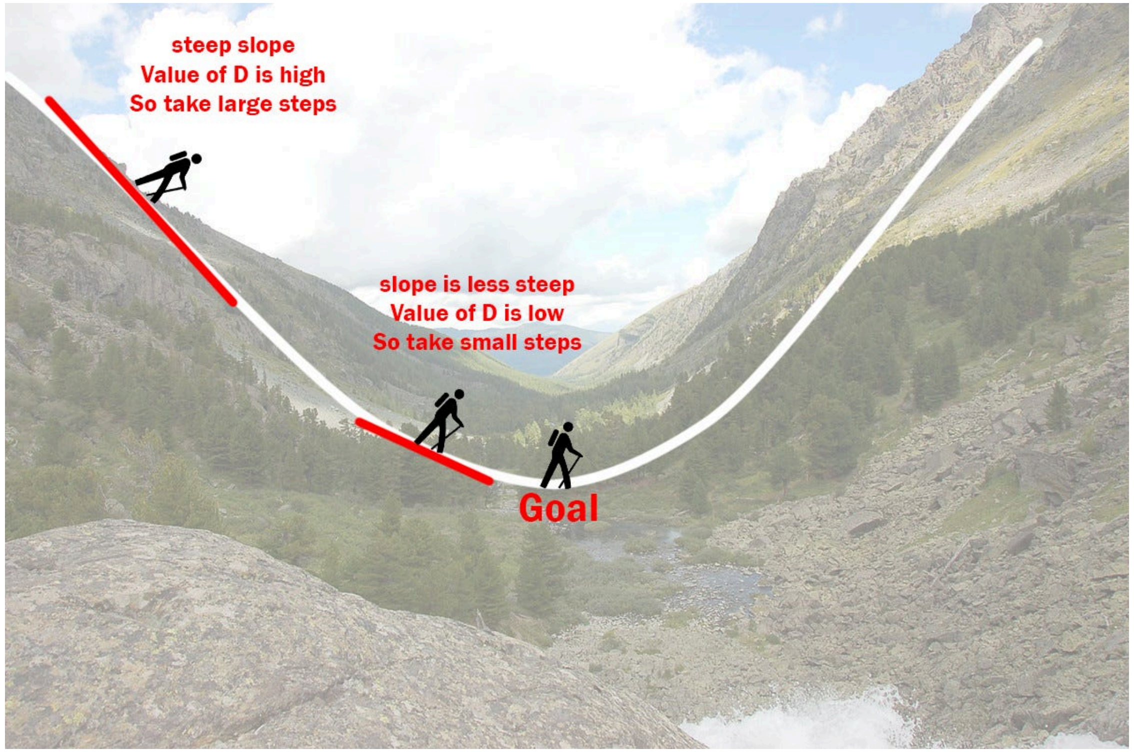
- **Goal**: Given a **differentiable** function $f(t)$, find the input t^* that minimizes $f(t)$.
- What does $\frac{d}{dt} f(t)$ mean?



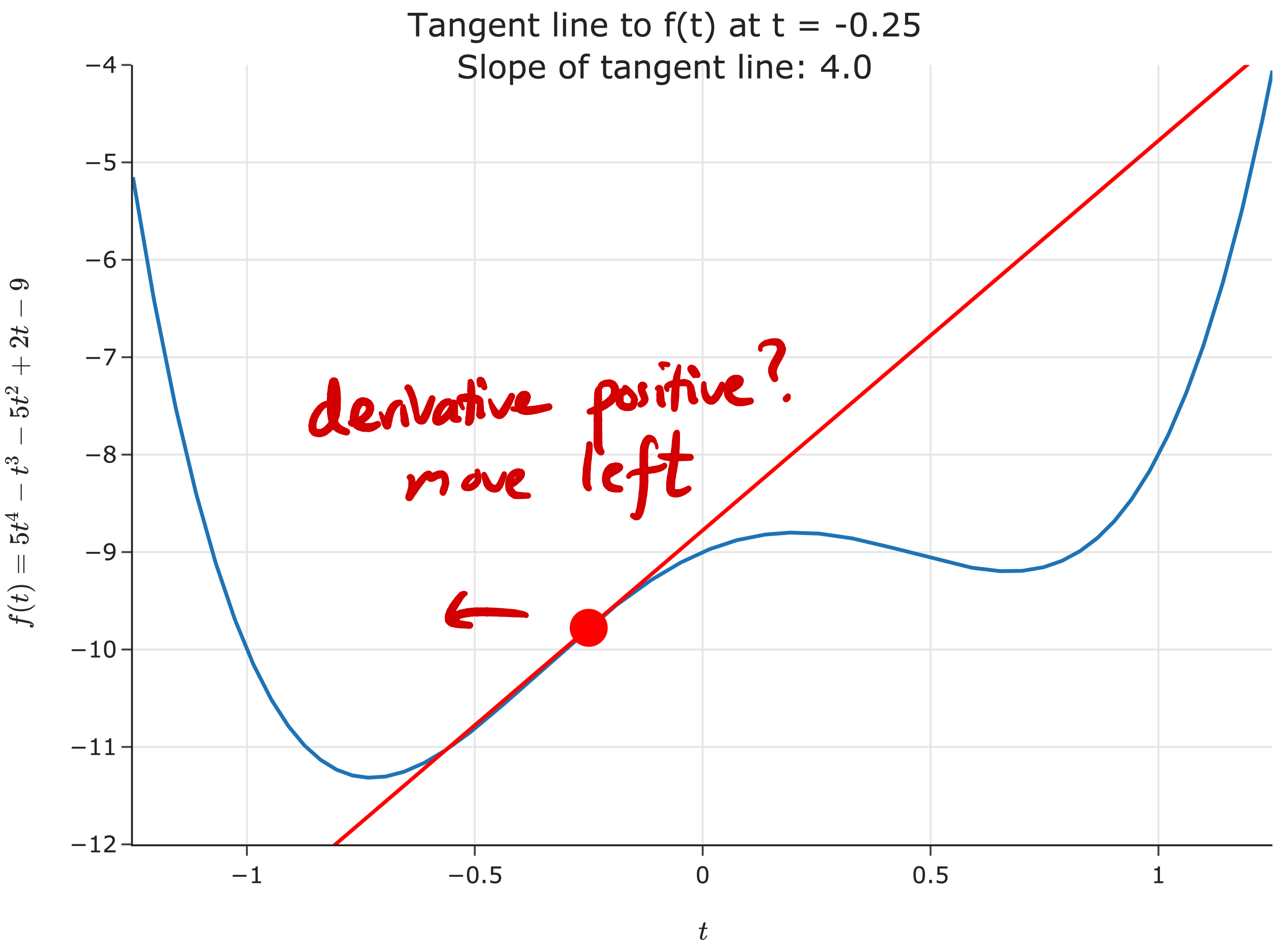
See dsc40a.com/resources/lectures/lec10 for an animated version of the previous slide!

Let's go hiking!

- Suppose you're at the top of a mountain ⛰️ and need to get **to the bottom**.
- Further, suppose it's really cloudy ☁️, meaning you can only see a few feet around you.
- **How** would you get to the bottom?



Searching for the minimum

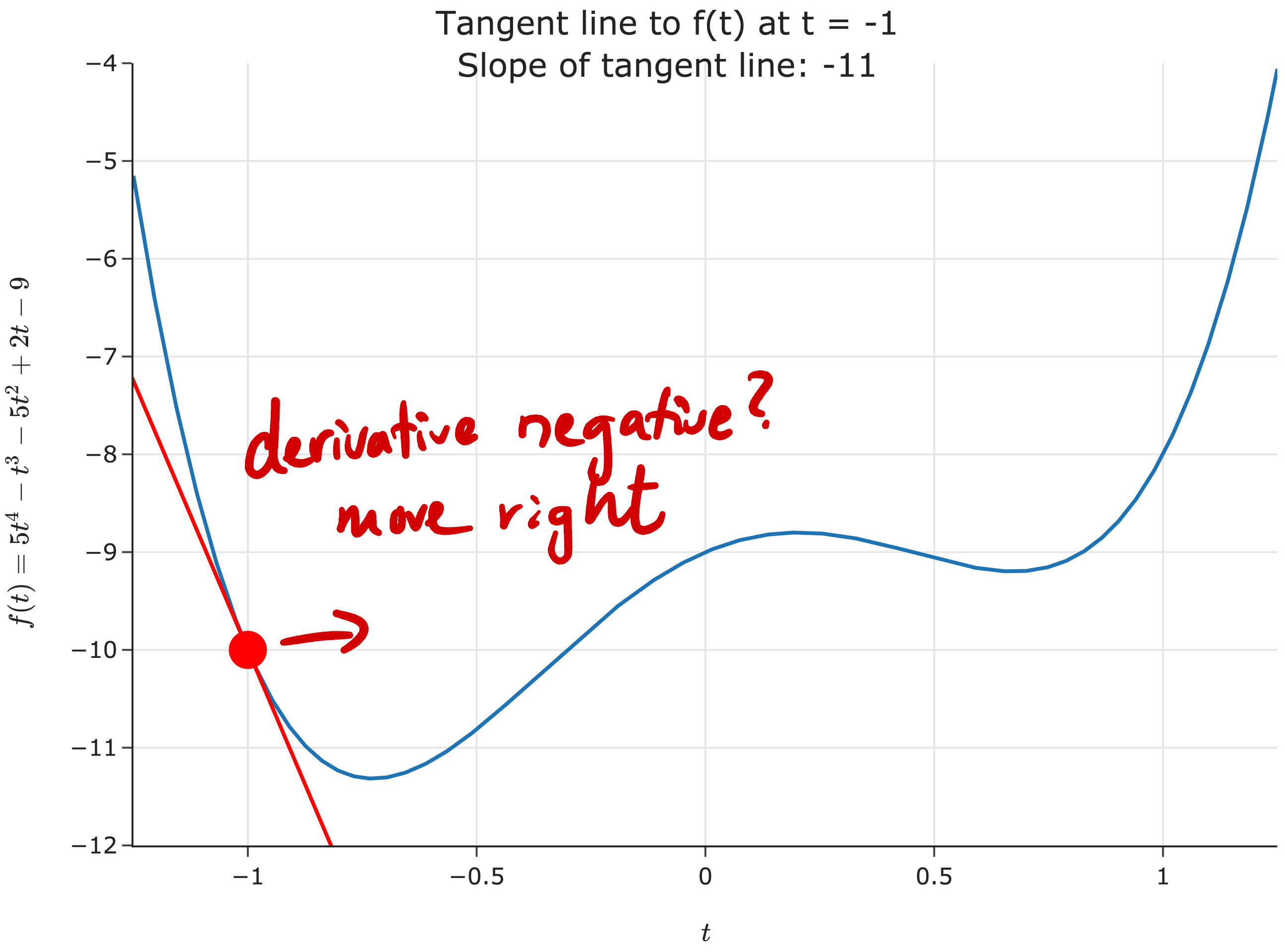


Suppose we're given an initial *guess* for a value of t that minimizes $f(t)$.

If the **slope of the tangent line at $f(t)$** is **positive** 📈:

- Increasing t **increases** f.
- This means the minimum must be to the **left** of the point $(t, f(t))$.
- Solution: **Decrease** t ⬇️.

Searching for the minimum



Suppose we're given an initial *guess* for a value of t that minimizes $f(t)$.

If the **slope of the tangent line at $f(t)$** is **negative** 📉:

- Increasing t **decreases** f.
- This means the minimum must be to the **right** of the point $(t, f(t))$.
- Solution: **Increase** t ⬆️.

$t_0, t_1, \ldots$: guesses for the t^* that minimizes $f(t)$.

Intuition

- To minimize $f(t)$, start with an initial guess t_0.
- Where do we go next?
    - If $\frac{df}{dt}(t_0) > 0$, **decrease** t_0.
    - If $\frac{df}{dt}(t_0) < 0$, **increase** t_0.
- One way to accomplish this:

if $\frac{df}{dt}(t_0) > 0$,

$$t_1 = t_0 - \square$$

if $\frac{df}{dt}(t_0) < 0$,

$$t_1 = t_0 + \square$$

$$t_1 = t_0 \underbrace{- \frac{df}{dt}(t_0)}_{\text{opposite the direction of the derivative!}}$$

when $\frac{df}{dt}(t_i)$ is small, we're closer to a minimum, and take smaller steps!

Gradient descent

To minimize a **differentiable** function f:

- Pick a positive number, α. This number is called the **learning rate**, or **step size**.
- Pick an **initial guess**, t_0.
- Then, repeatedly update your guess using the **update rule**:

$$t_{i+1} = t_i - \underbrace{\alpha \frac{df}{dt}(t_i)}_{\text{walking opposite the direction of the derivative}}$$

step size
- small : small steps
- big : big steps

- Repeat this process until **convergence** – that is, when t doesn't change much.
- This procedure is called **gradient descent**.

What is gradient descent?

- Gradient descent is a numerical method for finding the input to a function f that minimizes the function.
- Why is it called **gradient** descent?
  - The gradient is the extension of the derivative to functions of multiple variables.
  - We will see how to use gradient descent with multivariate functions next class.
- What is a **numerical** method?
  - A numerical method is a technique for approximating the solution to a mathematical problem, often by using the computer.
- Gradient descent is **widely used** in machine learning, to train models from linear regression to neural networks and transformers (includng ChatGPT)!

See dsc40a.com/resources/lectures/lec10 for animated examples of gradient descent, and see this notebook for the associated code!

Lingering questions

Next class, we'll explore the following ideas:

- When is gradient descent *guaranteed* to converge to a global minimum?
    - What kinds of functions work well with gradient descent?
- How do I choose a step size?
- How do I use gradient descent to minimize functions of multiple variables, e.g.:

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n}(y_i - (w_0 + w_1 x_i))^2$$

Gradient descent and empirical risk minimization

- While gradient descent can minimize other kinds of differentiable functions, its most common use case is in **minimizing empirical risk**.
- For example, consider:
  - The constant model, $H(x) = h$.
  - The dataset $-4, -2, 2, 4$.
  - The initial guess $h_0 = 4$ and the learning rate $\alpha = \frac{1}{4}$.
- **Exercise**: Find h_1 and h_2.