Lecture 13

Combinatorics

DSC 40A, Spring 2024

Announcements

- Homework 5 is due **tonight**.

Agenda

- Recap: The domino problem.
- Combinatorics.
- Lots and lots of examples.

Remember, we've posted **many** probability resources on the resources tab of the course website. These will come in handy! Specific resources you should look at:

- The DSC 40A probability roadmap, written by Janine Tiefenbruck.
- The textbook Theory Meets Data, which explains many of the same ideas and contains more practice problems.

For today's lectures specifically, there are two supplementary videos I created that you should watch. Both are linked in this playlist, which is also linked at dsc40a.com.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

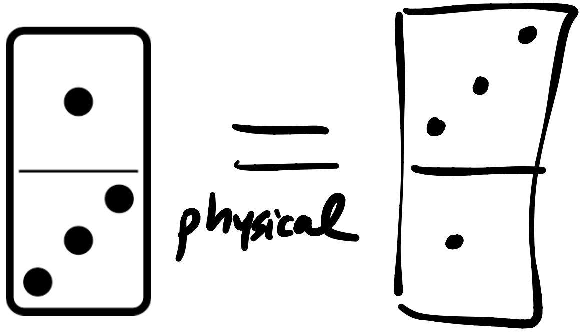
Recap: The domino problem

Example: Dominoes

(source: 538)

In a set of dominoes, each tile has two sides with a number of dots on each side: 0, 1, 2, 3, 4, 5, or 6. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.

7	6	5	4	3	2	1
(0,0)						
(0,1)	(1,1)					
(0,2)	(1,2)	(2,2)				
(0,3)	(1,3)	(2,3)	(3,3)			
(0,4)	(1,4)	(2,4)	(3,4)	(4,4)		
(0,5)	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	
(0,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)



$$1+2+\cdots+7 = \frac{7(\cancel{8}^{4})}{\cancel{2}} = 28$$

Example: Dominoes

Question 1: What is the probability of drawing a "double" from a set of dominoes – that is, a tile with the same number on both sides?

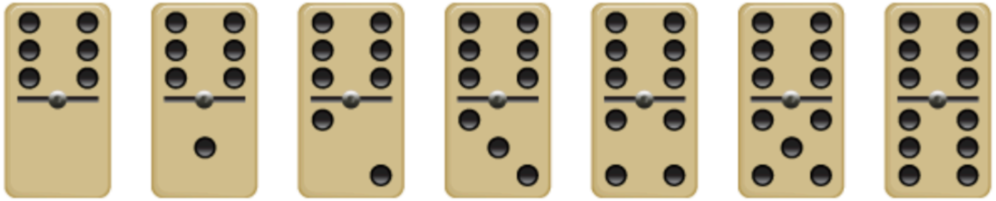
of the 28 tiles, 7 are doubles

$$\Rightarrow \mathbb{P}(\text{double}) = \frac{7}{28} = \boxed{\frac{1}{4}}$$

7	6	5	4	3	2	1
(0,0)						
(0,1)	(1,1)					
(0,2)	(1,2)	(2,2)				
(0,3)	(1,3)	(2,3)	(3,3)			
(0,4)	(1,4)	(2,4)	(3,4)	(4,4)		
(0,5)	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	
(0,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Example: Dominoes

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?



7	6	5	4	3	2	1
(0,0)						
(0,1)	(1,1)					
(0,2)	(1,2)	(2,2)				
(0,3)	(1,3)	(2,3)	(3,3)			
(0,4)	(1,4)	(2,4)	(3,4)	(4,4)		
(0,5)	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	
(0,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	$\boxed{(6,6)}$

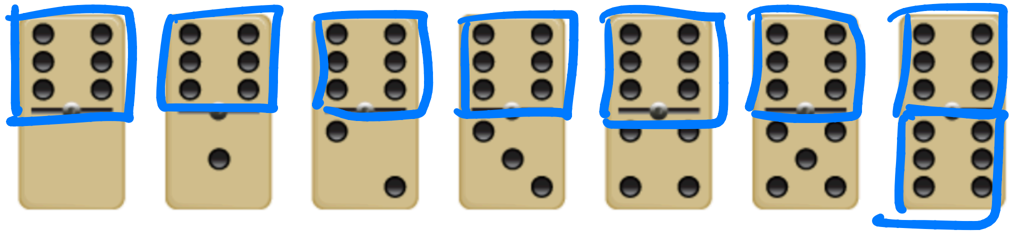
Friend picked one of these 7

$$\mathbb{P}(\text{double} \mid \text{at least one } 6) = \frac{\#\text{doubles w/ at least one } 6}{\#\text{ at least one } 6} = \boxed{\frac{1}{7}}$$

Example: Dominoes

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has 6 dots. What is the probability that this tile is a double, with 6 on both sides?

given that we picked one of these 8



special!

6 across from another 6

28 dominoes

$\Rightarrow 28 \times 2 = 56$ half dominoes

$\rightarrow$ of the 8 half-dominoes equal to 6, only 2 are across from another 6 $\Rightarrow$

$$\mathbb{P}(\text{double} \mid \text{uncovered a } 6) = \frac{2}{8} = \boxed{\frac{1}{4}}$$

||

equals $\mathbb{P}(\text{double})$ from 2 slides ago!

See 538's explanation here.

Don't believe me? Believe the simulation!

- To verify your answer to a probability problem, you can often run a simulation!
- This notebook has a simulation of the domino problem.

Probability of a double:

```
In [46]:  is_double / n
Out[46]:  0.250091
```

$\approx \frac{1}{4} = \mathbb{P}(\text{double})$

Probability of double 6s, given that at least one side is a 6:

```
In [47]:  double_6 / at_least_one_6
Out[47]:  0.14351561030844512
```

$\approx \frac{1}{7} = \mathbb{P}(\text{double} \mid \text{at least one 6})$

Probability of double 6s, given that we uncovered a single side and it was a 6:

```
In [48]:  double_6 / one_side_6
Out[48]:  0.25142644852869017
```

$\approx \frac{1}{4} = \mathbb{P}(\text{double} \mid \text{uncovered a 6})$

Combinatorics

Motivation

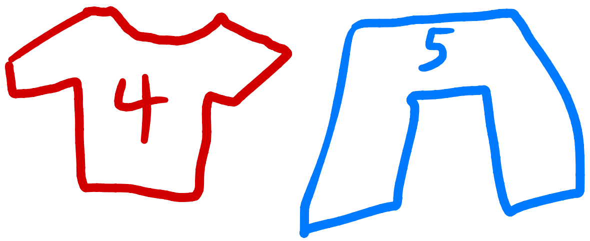
- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called **combinatorics**.

# of ways to do something

selecting k items from
n possibilities

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?



$\rightarrow 4 \cdot 5 = 20$ possible outfits

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement (i.e. repetition is allowed)**, such that **order matters.** $\rightarrow$ A12345678 $\neq$ A87654321 : these are different!
- **Example**: Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times. How many such sequences are there?

$n=52$
$k=4$

$\underset{\text{1st card}}{\underline{52}}\ \underset{\text{2nd}}{\underline{52}}\ \underset{\text{3rd}}{\underline{52}}\ \underset{\text{4th}}{\underline{52}} \rightarrow \boxed{52^4}$ sequences

- **Example**: A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

$n=10$
$k=8$

A $\underset{\text{1st digit}}{\underline{10}}\ \underset{\text{2nd}}{\underline{10}}\ \underset{\text{3rd}}{\underline{10}}\ \underset{\text{4th}}{\underline{10}}\ \underset{\text{5th}}{\underline{10}}\ \underset{\text{6th}}{\underline{10}}\ \underset{\text{7th}}{\underline{10}}\ \underset{\text{8th}}{\underline{10}} \rightarrow \underbrace{10 \times 10 \times \ldots \times 10}_{8\text{ times}}$ and

$= \boxed{10^8}$

Sequences

In general, the number of ways to select k elements from a group of n possible elements **with replacement (i.e. repetition is allowed)** and **order matters** is n^k.

$$\underbrace{n}_{1^{st}\text{ element}} \quad \underbrace{n}_{2^{nd}} \quad \cdots \quad \underbrace{n}_{(k-1)^{th}} \quad \underbrace{n}_{k^{th}} = \underbrace{n \times n \times \cdots \times n}_{k \text{ times}}$$

$$= \boxed{n^k}$$

don't jump straight to the formula!

$\Rightarrow$ write out the blanks.

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement (i.e. repetition is not allowed)**, such that **order matters**.
- **Example**: Draw 4 cards, without replacement, from a standard 52-card deck. How many such permutations are there?

could be anything, other than the first card!

$n=52$
$k=4$

$$\underbrace{52}_{1^{st}\text{ card}} \quad \underbrace{51}_{2^{nd}} \quad \underbrace{50}_{3^{rd}} \quad \underbrace{49}_{4^{th}} = 52\cdot 51\cdot 50\cdot 49$$

- **Example**: How many ways are there to select a president, vice president, and secretary from a group of 8 people?

answer

also the answer!

$n=8$
$k=3$

$$\underset{P}{\underline{8}} \quad \underset{VP}{\underline{7}} \quad \underset{S}{\underline{6}} = \boxed{8\cdot 7\cdot 6 = 336}$$

$$\underset{S}{\underline{8}} \quad \underset{P}{\underline{7}} \quad \underset{VP}{\underline{6}} = 8\cdot 7\cdot 6 = 336$$

Permutations

Example: $n=52$, $k=4$ multiply from n to $n-k+1$

$n=52$

$n-k+1 = 52-4+1 = 52-3 = 49$

$P(52,4) = 52 \cdot 51 \cdot 50 \cdot 49$

same as last slide! checks out.

- In general, the number of ways to select k elements from a group of n possible elements **without replacement (i.e. repetition is not allowed)** and **order matters** is:

$$\xrightarrow{\text{permutations}} P(n,k) = (n)(n-1)\ldots(n-k+1)$$

- To simplify: recall that the definition of $n!$ is:

$$n! = (n)(n-1)\ldots(3)(2)(1)$$

- Given this, we can write:

$$P(n,k) = \boxed{\frac{n!}{(n-k)!}} = \frac{(n)(n-1)\ldots\ldots(n-k+2)(n-k+1)\cancel{(n-k)!}}{\cancel{(n-k)!}}$$

$$= (n)(n-1)\ldots(n-k+1)$$

important!

same!

Aside:

$6!$

$= 6 \cdot 5!$

$= 6 \cdot 5 \cdot 4!$

$= 6 \cdot 5 \cdot 4 \cdot 3!$

Question 🤔

Answer at q.dsc40a.com

UCSD has 8 colleges. In how many ways can I rank my top 3 choices?

- A. 24. $8 \cdot 3$
- B. 336.
- C. 512. 8^3
- D. 6561. 3^8
- E. None of the above.

$$\underbrace{8}_{1st} \; \underbrace{7}_{2nd} \; \underbrace{6}_{3rd} = \frac{8!}{5!}$$

don't try to find 8! and 5!

Special case of permutations

$k=n$

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is:

$$\underset{1^{st}}{\underline{n}}\ \underset{2^{nd}}{\underline{n-1}}\ \underset{3^{rd}}{\underline{n-2}}\ \cdots\ \underset{(n-1)^{th}}{\underline{2}}\ \underset{n^{th}}{\underline{1}} = (n)(n-1)\cdots(2)(1) = \boxed{n!}$$

- This is consistent with the formula:

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

$0! = 1$ by definition

- **Followup**: How many ways are there to arrange n people in a circle? $\boxed{\text{general: } (n-1)!}$

Example $n=4$

A
D B
C
=
D
C A
B
=
C
B D
A
=
B
A C
D

$\Rightarrow$ # arrangements $= \frac{4!}{4} = 3!$

Circle problem

fix one person in place

$\boxed{\text{head}}$

$\underline{1}$ $\underline{n-1}$

$\underline{2}$ $\underline{n-2}$

$\cdots$ $\underline{n-3}$

$$\Rightarrow \#\text{arrangements} = (n-1)(n-2)\cdots(2)(1) = \boxed{(n-1)!}$$

$$= \frac{n!}{n}$$

rotations

Combinations

- A **combination** is a set of k elements selected from a group of n possible elements **without replacement (i.e. repetition is not allowed)**, such that **order does not matter**.
- **Example**: There are 4 ice cream flavors. In how many ways can you pick two different flavors?

C V S T

chocolate vanilla strawberry Thai Tea (at Sumi Sumi)

CV VS ST
CS VT
CT

$\Rightarrow$ 6 combinations

NOT the answer

$$\underset{\text{1st scoop}}{4} \times \underset{\text{2nd scoop}}{3} = \cancel{\boxed{12}}$$

double counting!

Consider the case where we select 3 flavors from a set of 4.

$$\underset{1^{st}}{\underline{4}} \quad \underset{2^{nd}}{\underline{3}} \quad \underset{3^{rd}}{\underline{2}} = \frac{24 \text{ permutations}}{3! \text{ permutations/combination}} = \frac{24}{6} = \boxed{4} \text{ combinations}$$

↑ adjustment

Consider one combination, VCT. There are 3! ways of rearranging it that are all the same combination!

VCT VTC CTV CVT TCV TVC

⇒ Solution: Divide 24 by 3! adjustment for repeated permutations

From permutations to combinations

order matters

- There is a close connection between:
  - the number of **permutations** of k elements selected from a group of n, and
  - the number of **combinations** of k elements selected from a group of n.

order doesn't matter

$$\#\ \text{combinations} = \frac{\#\ \text{permutations}}{\#\ \text{orderings of } k \text{ elements}}$$

adjustment of $k!$

- Since $\#\ \text{permutations} = \frac{n!}{(n-k)!}$ and $\#\ \text{orderings of } k \text{ elements} = k!$, we have:

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

adjustment

Combinations

In general, the number of ways to select k elements from a group of n elements **without replacement (i.e. repetition is not allowed) and **order does not matter** is:

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!k!}}$$

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

$\binom{n}{k}$ not a fraction!

binomial theorem

$(a+b)^n = \cdots$

Math 183, Econ 120A

permutation : AGR $\neq$ GRA
P VP S   P VP S

Example: Committees

combination : AGR $=$ GRA
no roles

- How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$$\underset{P}{\underline{8}}\ \underset{VP}{\underline{7}}\ \underset{S}{\underline{6}} = 8 \cdot 7 \cdot 6 = 336$$

- How many ways are there to select a committee of 3 people from a group of 8 people?

$$\frac{336}{3!} = \binom{8}{3}$$

- If you're ever confused about the difference between permutations and combinations, **come back to this example**.

- More generally, don't jump straight to a formula: think about what the question is asking for.

Aside: Simplifying $\binom{n}{k}$

It's true that:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

However, when asked to simplify the value of $\binom{n}{k}$, do so strategically!

Example

$$\binom{16}{3} = \frac{16!}{3!\,13!} = \frac{16 \cdot 15 \cdot 14 \cdot \cancel{13!}}{3!\,\cancel{13!}}$$

$$= \frac{\overset{8}{\cancel{16}} \cdot \overset{5}{\cancel{15}} \cdot 14}{\cancel{3} \cdot \cancel{2}}$$

$$= 8 \cdot 5 \cdot 14$$

$$= 40 \cdot 14 = \boxed{560}$$

Question 🤔

Answer at q.dsc40a.com

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in a set of dominoes?

- A. $\binom{7}{2}$
- B. $\binom{7}{1} + \binom{7}{2}$
- C. $P(7, 2)$
- D. $\frac{P(7,2)}{P(7,1)} 7!$

2 cases:

double

$\binom{7}{1} = 7$

two unique halves

$\binom{7}{2} = \frac{7 \cdot \cancel{6}^{3}}{\cancel{2}} = 21$

7 options (0, 1, 2, 3, 4, 5, 6), choose 2

Total: $\binom{7}{1} + \binom{7}{2}$

$= 7 + 21 = \boxed{28}$

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	No, order doesn't matter
With replacement Repetition allowed	$\boxed{n^k}$ possible **sequences**	more complicated: watch this video* domino example!
Without replacement Repetition not allowed	$\boxed{\frac{n!}{(n-k)!}}$ possible **permutations**	$\boxed{\binom{n}{k}}$ **combinations**

**or see the previous slide.*

the slides for the "Even more examples" section at the end are annotated!

More examples

All we're going to do for the remainder of today's lecture and much of Tuesday's lecture is work through examples of combinatorics problems.

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $\mathbb{P}(A) = \frac{|A|}{|S|}$. the number of outcomes in A
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip**: Before starting a probability problem, always think about what the sample space S is!

Overview: Selecting students

We're going answer the same question using several different techniques.

Avocado

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random without replacement. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 1: Using permutations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$S = \{$ all permutations of 5 students selected from 20 $\}$

$$\text{denominator} = |S| = \frac{20!}{15!}$$

$$\underset{1^{st}}{\underline{20}} \quad \underset{2^{nd}}{\underline{19}} \quad \underset{3^{rd}}{\underline{18}} \quad \underset{4^{th}}{\underline{17}} \quad \underset{5^{th}}{\underline{16}}$$

$S = \{$ all permutations of 5 students selected from 20 $\}$

denominator $= |S| = \frac{20!}{15!}$

$\underline{20}$ $\underline{19}$ $\underline{18}$ $\underline{17}$ $\underline{16}$

1st 2nd 3rd 4th 5th

numerator = # permutations that include Avi

5 cases :

$\boxed{A}$	19	18	17	16
19	$\boxed{A}$	18	17	16
19	18	$\boxed{A}$	17	16
19	18	17	$\boxed{A}$	16
19	18	17	16	$\boxed{A}$

$19 \cdot 18 \cdot 17 \cdot 16 \cdot \underbrace{5}_{\text{cases}}$

$$\mathbb{P} = \frac{\cancel{19 \cdot 18 \cdot 17 \cdot 16} \cdot 5}{20 \cdot \cancel{19 \cdot 18 \cdot 17 \cdot 16}} = \frac{5}{20} = \boxed{\frac{1}{4}}$$

next class

Selecting students

Method 2: Using permutations and the complement

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

next class

Selecting students

Method 3: Using combinations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

next class

Selecting students

Method 4: The "easy" way

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

next class

Question 🤔

Answer at q.dsc40a.com

With vs. without replacement

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

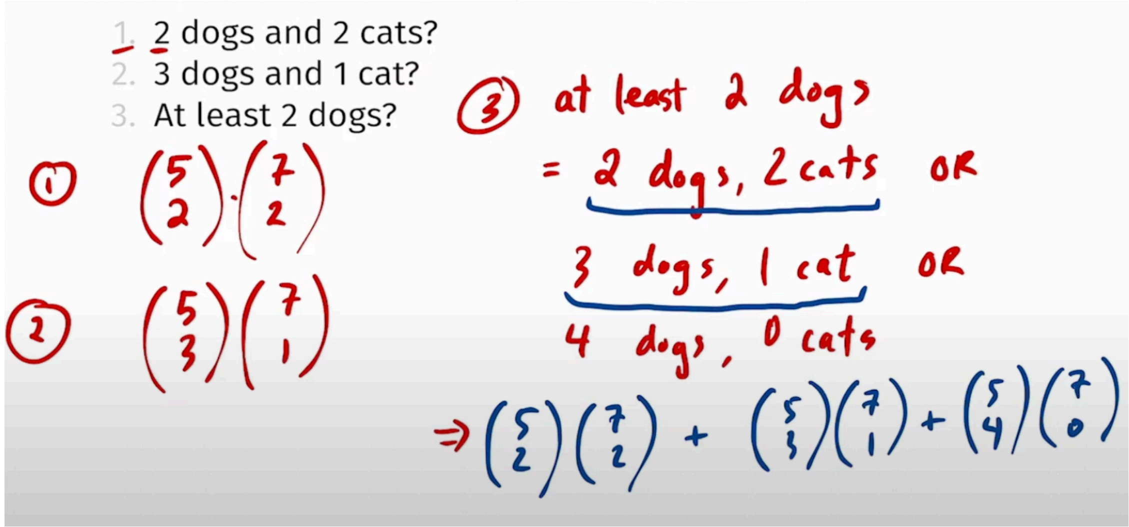
- A. Equal to.
- B. Greater than.
- C. Less than.

won't cover in lecture:
watch the video!

Even more examples

Watch the video linked below!

The following slides **will not** be covered in lecture. Instead, they're covered in this walkthrough video.



A screenshot from the walkthrough video.

We'll still cover more examples on Tuesday, but now you have a few more examples to refer to when working on Homework 6 over the weekend.

Example: Pets

Part 1: We have 12 pets: 5 dogs and 7 cats. In how many ways can we select 4 pets?

order doesn't matter

$$\binom{12}{4}$$

we don't care about the difference between dogs and cats, so we "choose" them together

Example: Pets

Part 2: We have 12 pets: 5 dogs and 7 cats. In how many ways can we select 4 pets such that we have...

1. 2 dogs and 2 cats?
2. 3 dogs and 1 cat?
3. At least 2 dogs?

① $\binom{5}{2} \cdot \binom{7}{2}$

5 dogs, choose 2
AND
7 cats, choose 2

② $\binom{5}{3}\binom{7}{1}$

③ at least 2 dogs = $\underbrace{\text{2 dogs, 2 cats}}$ OR 3 dogs, 1 cat OR 4 dogs, 0 cats

$$= \binom{5}{2}\binom{7}{2} + \binom{5}{3}\binom{7}{1} + \binom{5}{4}\binom{7}{0}$$

Example: Pets

Part 3: We have 12 pets: 5 dogs and 7 cats. We randomly select 4 pets. What's the probability that we selected at least 2 dogs?

$S = \{\text{all combinations of 4 pets}\}$

$$\mathbb{P}(\text{at least 2 dogs}) = \frac{\#\text{ combinations of 4 pets w/at least 2 dogs}}{\#\text{ combinations of 4 pets}}$$

$$= \frac{\binom{5}{2}\binom{7}{2} + \binom{5}{3}\binom{7}{1} + \binom{5}{4}\binom{7}{0}}{\binom{12}{4}}$$

from the previous slide!

Example: A fair coin

Suppose we flip a fair coin 10 times.

$\mathbb{P}(\text{heads}) = \mathbb{P}(\text{tails}) = \frac{1}{2}$

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

① $\underbrace{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)}_{10 \text{ times}} = \left(\frac{1}{2}\right)^{10}$ ← equivalently, there are 2^{10} sequences of 10 heads/tails, and this is 1 of them: $\boxed{\frac{1}{2^{10}}}$

② First, we determine the number of ways of seeing 5 heads and 5 tails

→ 10 flips, choose 5 to be heads: $\binom{10}{5}$

$$\mathbb{P}(5 \text{ heads}, 5 \text{ tails}) = \frac{\#\text{ sequences with } 5H, 5T}{\#\text{ sequences of } 10 \text{ flips}} = \frac{\binom{10}{5}}{2^{10}} = \binom{10}{5} \underbrace{\mathbb{P}\left(\begin{array}{c}\text{one sequence of} \\ 5 \text{ heads}, 5 \text{ tails}\end{array}\right)}$$

Example: An unfair coin

$\mathbb{P}(\text{heads}) = \frac{1}{3}$, $\mathbb{P}(\text{tails}) = \frac{2}{3}$

Suppose we flip an **unfair** coin 10 times. The coin is biased such that for each flip, $\mathbb{P}(\text{heads}) = \frac{1}{3}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

① $$\underset{T}{\left(\tfrac{2}{3}\right)}\underset{H}{\left(\tfrac{1}{3}\right)}\underset{T}{\left(\tfrac{2}{3}\right)}\underset{T}{\left(\tfrac{2}{3}\right)}\underset{H}{\left(\tfrac{1}{3}\right)}\underset{T}{\left(\tfrac{2}{3}\right)}\underset{H}{\left(\tfrac{1}{3}\right)}\underset{H}{\left(\tfrac{1}{3}\right)}\underset{T}{\left(\tfrac{2}{3}\right)}\underset{H}{\left(\tfrac{1}{3}\right)} = \underbrace{\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right)^5}$$

② $$\mathbb{P}(5 \text{ heads}, 5 \text{ tails}) = \#\,\text{sequences of } 5 \text{ heads}, 5 \text{ tails} \times \mathbb{P}(\text{one sequence of } 5 \text{ heads}, 5 \text{ tails})$$

$$= \binom{10}{5}\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right)^5$$