Lecture 14

More Combinatorics Examples

DSC 40A, Spring 2024

Announcements

- Homework 6 is due on **Thursday at 11:59PM**.
    - Combinatorics is hard! Come to office hours.

Agenda

- Example: Selecting students.
- Example: An unfair coin.
- Example: Poker.
- Example: Chess.

Remember, we've posted **many** probability resources on the resources tab of the course website. These will come in handy! Specific resources you should look at:

- The DSC 40A probability roadmap, written by Janine Tiefenbruck.
- The textbook Theory Meets Data, which explains many of the same ideas and contains more practice problems.

For combinatorics specifically, there are two supplementary videos I created that you should watch. Both are linked in this playlist, which is also linked at dsc40a.com.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

Example: Selecting students

We'll start by working through a few examples we didn't get to finish on Thursday.

Combinatorics = "counting"

sequences permutations combinations

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	**No, order doesn't matter**
With replacement Repetition allowed	$\boxed{n^k}$ possible **sequences**	more complicated: watch **this video**, or see the domino example
Without replacement Repetition not allowed	$\boxed{\frac{n!}{(n-k)!}}$ possible **permutations**	$\boxed{\binom{n}{k}}$ **combinations**

Using the binomial coefficient, $\binom{n}{k}$

As an example, let's evaluate:

$$\binom{9}{2} + \binom{9}{3}$$

$9 \cdot 8 \cdot 7!$ $\quad$ $9 \cdot 8 \cdot 7 \cdot 6!$

$$= \frac{9!}{7!\,2!} + \frac{9!}{6!\,3!} = \frac{9 \cdot \overset{4}{\cancel{8}} \cdot \cancel{7!}}{\cancel{7!}\,\cancel{2}!} + \frac{\overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot 7}{\cancel{3} \cdot \cancel{2}}$$

$$= 36 + 84 = \boxed{120}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$P(n,k)$

$$= \binom{n}{n-k}$$

e.g. $\binom{7}{2} = \binom{7}{5}$

choosing 2 to take

choosing 5 to not take

Aside: Combinatorial proof

We showed that $\binom{9}{2} + \binom{9}{3} = 120$.

It turns out that $\binom{10}{3} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4}}{\cancel{3} \cdot \cancel{2}} = 120.$

Can we argue why $\binom{9}{2} + \binom{9}{3} = \binom{10}{3}$?

Example: Coin flipping. How many ways can we see 3 heads in 10 flips of a coin? $\Rightarrow \binom{10}{3}$

Equivalent: $\underbrace{HHHTTTTTTT}_{\text{number of permutations}} = \frac{10!}{3!\,7!} = \binom{10}{3}$

\# ways to rearrange 3 heads

Proof of $\binom{9}{2}+\binom{9}{3}=\binom{10}{3}$.

Claim: Both sides count # of ways to see 3 heads in 10 flips.

Left side: consider 2 cases

1. First flip heads

   $\boxed{H}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_ \quad = \binom{9}{2}$

   fill in 2H, 7T

2. First flip tails

   $\boxed{T}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_ \quad = \binom{9}{3}$

   fill in 3H, 6T

Key idea: both $\binom{9}{2}+\binom{9}{3}$ and $\binom{10}{3}$ count the same thing, so they must be equal! "Combinatorial proof"

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $\mathbb{P}(A) = \frac{|A|}{|S|}$.

  $S = \{\text{set of all possible outcomes}\}$

- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.

- **Tip**: Before starting a probability problem, always think about what the sample space S is!

Overview: Selecting students

We're going answer the same question using several different techniques.

Avocado

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random without replacement. What is the probability that Avi is among the 5 selected students?

can't think about sequences

all students are equally likely to be chosen

weren't told whether order matters:
both permutations or combinations
give the same results!

Selecting students

Method 1: Using permutations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$S = \{$ all permutations of 5 students selected from 20 $\}$

To find $|S|$: $\underset{1^{st}}{\underline{20}}\ \underset{2^{nd}}{\underline{19}}\ \underset{3^{rd}}{\underline{18}}\ \underset{4^{th}}{\underline{17}}\ \underset{5^{th}}{\underline{16}} = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

To find # WITH Avi: 5 <u>cases</u>

$\boxed{A}\ \underline{19}\ \underline{18}\ \underline{17}\ \underline{16} + \underline{19}\ \boxed{A}\ \underline{18}\ \underline{17}\ \underline{16} + \underline{19}\ \underline{18}\ \boxed{A}\ \underline{17}\ \underline{16} + \underline{19}\ \underline{18}\ \underline{17}\ \boxed{A}\ \underline{16} + \underline{19}\ \underline{18}\ \underline{17}\ \underline{16}\ \boxed{A}$

$\Rightarrow$ permutations with Avi $= 5 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

$$\mathbb{P}(\text{Avi selected}) = \frac{\#\text{ permutations including Avi}}{\#\text{ total permutations}}$$

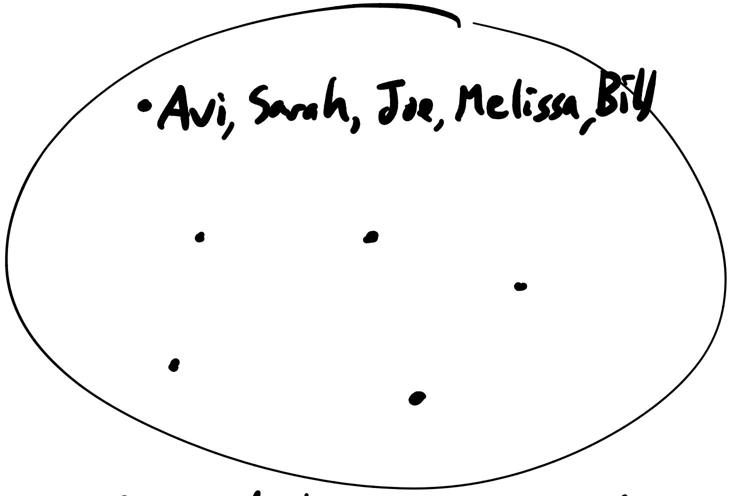
$$= \frac{5\cdot\cancel{19\cdot 18\cdot 17\cdot 16}}{20\cdot\cancel{19\cdot 18\cdot 17\cdot 16}}$$

$$= \frac{5}{20}$$

$$= \boxed{\frac{1}{4}}$$

Remember, $S = \{\text{all permutations}\}$

$S =$



permutations: $20\cdot 19\cdot 18\cdot 17\cdot 16$ of them

Selecting students

Method 2: Using permutations and the complement

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

We want $\mathbb{P}(\text{Avi selected})$

$\Rightarrow$ Idea: Start with $\mathbb{P}(\text{Avi not selected})$, then find $\mathbb{P}(\text{Avi selected}) = 1 - \mathbb{P}(\text{Avi not selected})$

$$\mathbb{P}(\text{Avi not selected}) = \frac{\#\text{ permutations WITHOUT Avi}}{\#\text{ total permutations}} = \frac{19\cdot\cancel{18}\cdot\cancel{17}\cdot\cancel{16}\cdot 15}{20\cdot\cancel{19}\cdot\cancel{18}\cdot\cancel{17}\cdot 16} = \frac{15}{20} = \frac{3}{4}$$

$\underset{1^{st}}{19}\ \ \underset{2^{nd}}{18}\ \ \underset{3^{rd}}{17}\ \ \underset{4^{th}}{16}\ \ \underset{5^{th}}{15}$

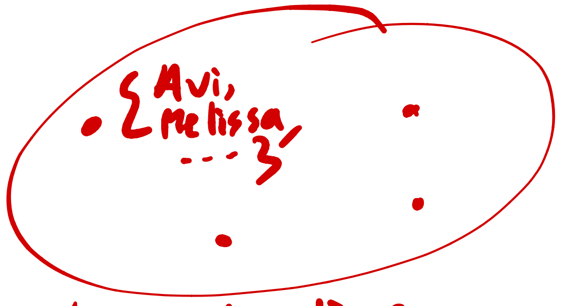
$$\boxed{\Rightarrow \mathbb{P}(\text{Avi selected}) = 1 - \frac{3}{4} = \frac{1}{4}}$$

benefit: didn't need 5 cases! easier :)

Selecting students

Method 3: Using combinations

S :



There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = { all combinations of 5 students selected from 20 }

$|S| = \binom{20}{5}$ ← the number of ways to choose 5 students from 20 such that order doesn't matter

A = { all combinations of 5 including Avi }

$|A| = \binom{19}{4}$ ← Avi is already included: choose 4 of the remaining 19

Benefit of combinations: didn't need 5 cases!

$$\mathbb{P}(\text{Avi selected}) = \frac{\#\text{ combinations with Avi}}{\#\text{ total combinations}}$$

$$= \frac{\binom{19}{4}}{\underbrace{\binom{20}{5}}} = \frac{19!}{4!\,\cancel{15!}} \cdot \frac{5!\,\cancel{15!}}{20!}$$

reciprocal of $\binom{20}{5}$

$$= \frac{19!\,5!}{4!\,20!} = \frac{\cancel{19!}\,5\cdot\cancel{4!}}{\cancel{4!}\,20\cdot\cancel{19!}}$$

$$= \frac{5}{20}$$

$$= \boxed{\frac{1}{4}}$$

We could use either permutations OR combinations because we weren't told if order matters!

Notice : $\mathbb{P}(\text{Avi selected}) = \frac{5}{20} = \frac{1}{4}$

Selecting students

Method 4: The "easy" way

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Thought experiment: shuffle all 20 students in a line, select the first 5

5 "good" positions

$S = \{\text{all possible positions Avi could be in}\} \Rightarrow |S| = 20$

$$\mathbb{P}(\text{Avi selected}) = \frac{\#\text{ "good" positions}}{\#\text{ total positions}} = \frac{5}{20} = \boxed{\frac{1}{4}}$$

without rep: $\mathbb{P}$(Avi selected) $= \frac{1}{4} = 25\%$.

with rep: $\mathbb{P}$(Avi selected) $= \mathbb{P}$(Avi selected at least once)

$= 1 - \mathbb{P}$(Avi never selected)

$= 1 - \left(\frac{19}{20}\right)^5 \approx 22\%$

Question 🤔

Answer at q.dsc40a.com

With vs. without replacement

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

- A. Equal to.
- B. Greater than.
- C. Less than.

without replacement: pool gets smaller after every selection, increasing the chance Avi is picked

$$\underbrace{\frac{1}{20} \to \frac{1}{19} \to \frac{1}{18} \to \frac{1}{17} \to \frac{1}{16}}_{\text{getting bigger}}$$

vs. with replacement: all $\frac{1}{20}$

$\Rightarrow$ with replacement, could have duplicates; without, all 5 are unique.

Example: An unfair coin

Followup: An unfair coin

In the video you were asked to watch, we flipped an **unfair** coin 10 times, where the coin was biased such that for each flip, $\mathbb{P}(\text{heads}) = \frac{1}{3}$. → assume each flip independent

1. What is the probability that we see the specific sequence HHHHTTTTTT?

$$\mathbb{P}(HHHHTTTTTT) = \left(\tfrac{1}{3}\right) \cdot \left(\tfrac{1}{3}\right) \cdot \left(\tfrac{1}{3}\right) \cdot \left(\tfrac{1}{3}\right) \cdot \left(\tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}\right)$$

and

$$= \boxed{\left(\tfrac{1}{3}\right)^4 \left(\tfrac{2}{3}\right)^6}$$

2. What is the probability that we see exactly 4 heads?

Many such sequences: HHHHTTTTTT, HTHTHTHTTT, TTHHHHTTTT, ...

ways to arrange 4H, 6T: $\binom{10}{4}$

$$\mathbb{P}(4\ H,\ 6\ T) = \#\text{ways of seeing } 4\ H,\ 6\ T \times \mathbb{P}(\text{one such sequence}) = \boxed{\binom{10}{4} \left(\tfrac{1}{3}\right)^4 \left(\tfrac{2}{3}\right)^6}$$

Followup: An unfair coin

3. What is the probability that we see exactly k heads, where $0 \leq k \leq 10$?

$$\mathbb{P}(k \text{ heads}, 10-k \text{ tails}) = \boxed{\binom{10}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{10-k}}$$

ways to arrange k heads, 10-k tails

4. What is the probability that we see *at least* k heads, where $0 \leq k \leq 10$?

$$\mathbb{P}(\text{at least } k \text{ heads}) = \mathbb{P}(k \text{ heads}) + \mathbb{P}(k+1 \text{ heads}) + \dots + \mathbb{P}(10 \text{ heads})$$

$$= \boxed{\sum_{i=k}^{10}\binom{10}{i}\left(\frac{1}{3}\right)^i\left(\frac{2}{3}\right)^{10-i}}$$

Example: Poker

Deck of cards

- There are **52** cards in a standard deck.
- Each card has a suit (4 possibilities) and a value (13 possibilities). "face values"

hearts ♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
diamonds ♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
clubs ♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
spades ♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Deck of cards

1. How many 5 card hands are there in poker?

52 possible, choose 5 $\Rightarrow$ order doesn't matter!

$\boxed{\binom{52}{5}}$

$\underline{52}\ \underline{51}\ \underline{50}\ \underline{49}\ \underline{48} \Rightarrow \dfrac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5!}$

this assumes order matters,
but it doesn't!

2. How many 5 card hands are there where all cards are of the same suit (i.e. a "flush")?

① all hearts $\binom{13}{5}$

② all diamonds $\binom{13}{5}$

③ all clubs $\binom{13}{5}$

④ all spades $\binom{13}{5}$

$\# = \boxed{4\cdot\binom{13}{5}}$

$\binom{4}{1}$: "4 suits, choose 1"

13 values, choose 5

Deck of cards

3. How many 5 card hands are there that include four cards of the same value (i.e. a "four-of-a-kind")?

aaaab

52 − 4

① four 2s : 2, 2, 2, 2, $\underline{48}$

② four 3s : 3, 3, 3, 3, $\underline{48}$

⋮

⑬ four As : A, A, A, A, $\underline{48}$

$$\# = \boxed{13 \cdot 48}$$

face values

Deck of cards

4. How many 5 card hands are there that have all card values consecutive (i.e. a "straight")? Assume A is the highest

9 cases/starting points

2-6
3-7
4-8 → i.e. 4, 5, 6, 7, 8
5-9
6-10
7-J
8-Q
9-K
10-A

after picking a starting point, for each card, 4 possible suits

$$\underline{4}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{4}$$

lowest → highest

$$\# = \boxed{9 \cdot 4^5}$$

starting points

ways of selecting suits

Deck of cards

5. How many 5 card hands are there that have all card values consecutive and of the same suit (i.e. a "straight flush")?

Same 9 starting points as on the last slide

BUT

all 5 cards need the same suit!

4 possible suits:

$$\# = \boxed{\underbrace{9}_{\#\text{ starting points}} \cdot \underbrace{4}_{\#\text{ suits}} = 36}$$

Deck of cards

6. How many 5 card hands are there that include exactly one pair of values (e.g. aabcd)?

aabcd

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$$

choosing the value to duplicate

choosing the suits of the duplicated value

choose 3 different values

4 options for each suit

NOT $\binom{48}{3}$: this could have another pair! :(

see practice site for additional explanations

Example: Chess

Chess pieces

on practice site!

(Source: Spring 2023 Midterm 2, Problem 3)

A set of chess pieces has 32 pieces. 16 of these are black and 16 of these are white. **In each color**, the 16 pieces are:

8 pawns, 2 bishops, 2 knights, 2 rooks, 1 queen, and 1 king

When there are multiple pieces of a given color and type (for example, 8 white pawns), we will assume they are **indistinguishable** from one another.

In this problem, a **lineup** is a way of arranging items in a straight line.

Chess pieces

1. A chess player lines up all 16 **white pieces** from the set of chess pieces. How many different-looking lineups can be created? Remember, some pieces look the same.

① 8 pawns
2 bishops
2 knights
2 rooks
1 king
1 queen

$$\boxed{\frac{16!}{8!\,2!\,2!\,2!}}$$

total pieces

repeat pawns

repeat bishops

equal!

② 16 positions, choose 8 to fill with pawns

$$\boxed{\binom{16}{8}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1}}$$

of the 8 remaining positions, fill 2 with bishops

of the remaining 6 positions, fill 2 with knights

⋮

Chess pieces

2. A chess player lines up all 16 **pawns** from the set of chess pieces. How many lineups have white pawns on both ends?

fill with 14 pieces:
6 W, 8 B

$\boxed{W}$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ $\boxed{W}$

$\boxed{\binom{14}{6}}$: 14 positions, choose 6 to fill with white

$= \binom{14}{8}$

Chess pieces

3. A chess player lines up all 16 **pawns** from the set of chess pieces. Assuming that each different-looking lineup is equally likely, what is the probability that the lineup has two of the same-colored pawns on both ends (both black or both white)?

$\boxed{W}$ ______ $\boxed{W}$: $\binom{14}{6}$

$\boxed{B}$ ______ $\boxed{B}$: $\binom{14}{6}$

$$\mathbb{P}(\text{same colored pawns on ends}) = \frac{2\binom{14}{6}}{\binom{16}{8}} = \frac{2\cdot 14!}{6!\,\cancel{8!}}\cdot\frac{\cancel{8!}\,8!}{16!} = \frac{\cancel{2}\cdot\cancel{14!}\cdot\cancel{8}\cdot 7\cdot\cancel{6!}}{\cancel{6!}\cdot\cancel{16}_{2}\cdot 15\cdot\cancel{14!}} = \boxed{\frac{7}{15}}$$

16 positions, choose 8 to be white

Summary

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

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Without replacement Repetition not allowed	$\boxed{\frac{n!}{(n-k)!}}$ possible **permutations**	$\boxed{\binom{n}{k}}$ **combinations**