Lecture 16

# Independence and Conditional Independence

DSC 40A, Spring 2024

## Announcements

- There is no live lecture today (Tuesday). Instead, the lecture video will be pre-recorded and posted on the course website by Tuesday morning.
  - There's also a lecture note I wrote for this lecture that you should read.
- Homework 7 is due on **Thursday at 11:59PM**.
- The final exam is soon: start practicing at practice.dsc40a.com!
  - There are tons of past probability exams, searchable by topic.

## Agenda

- Independence.
- Conditional independence.

Remember, we've posted **many** probability resources on the resources tab of the course website. These will come in handy! Specific resources you should look at:

- The DSC 40A probability roadmap, written by Janine Tiefenbruck.
- The textbook Theory Meets Data, which explains many of the same ideas and contains more practice problems.

**For combinatorics specifically, there are two supplementary videos I created that you should watch. Both are linked in this playlist, which is also linked at dsc40a.com.**

## Question 🤔

Answer at q.dsc40a.com

**Remember, you can always ask questions at q.dsc40a.com!**

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

# Independence

## Updating probabilities

- Bayes' Theorem describes how to update the probability of one event, given that another event has occurred.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(A)}$$

  - $\mathbb{P}(B)$ can be thought of as the "prior" probability of $B$ occurring, before knowing anything about $A$.
  - $\mathbb{P}(B|A)$ is sometimes called the "posterior" probability of $B$ occurring, given that $A$ occurred.

- What if knowing that $A$ occurred doesn't change the probability that $B$ occurs? In other words, what if:

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

## Independent events

- $A$ and $B$ are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$\mathbb{P}(B|A) = \mathbb{P}(B) \qquad\qquad \mathbb{P}(A|B) = \mathbb{P}(A)$$

- Otherwise, $A$ and $B$ are **dependent events**.
- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

## Independent events

- **Equivalent definition**: $A$ and $B$ are independent events if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- To check if $A$ and $B$ are independent, use whichever is easiest:
  - $\mathbb{P}(B|A) = \mathbb{P}(B)$.
  - $\mathbb{P}(A|B) = \mathbb{P}(A)$.
  - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

# Question 🤔

**Answer at q.dsc40a.com**

### Mutual exclusivity and independence

Suppose $A$ and $B$ are two events with non-zero probabilities. Is it possible for $A$ and $B$ to be both mutually exclusive and independent?

- A. Yes.
- B. No.

## Example: Venn diagrams

For three events $A$, $B$, and $C$, we know that:

- $A$ and $C$ are independent,
- $B$ and $C$ are independent,
- $A$ and $B$ are mutually exclusive,
- $\mathbb{P}(A \cup C) = \frac{2}{3}$, $\mathbb{P}(B \cup C) = \frac{3}{4}$, $\mathbb{P}(A \cup B \cup C) = \frac{11}{12}$.

Find $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(C)$.

## Example: Cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - $A$ is the event that the first card is a heart.
  - $B$ is the event that the second card is a club.
- If you draw the cards **with** replacement, are $A$ and $B$ independent?
- If you draw the cards **without** replacement, are $A$ and $B$ independent?

## Example: Cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw one card from a deck of 52.
    - $A$ is the event that the card is a heart.
    - $B$ is the event that the card is a face card (J, Q, K).
- Are $A$ and $B$ independent?

## Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

## Example: Breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

# Conditional independence

## Conditional independence

- Sometimes, events that are dependent **become** independent, upon learning some new information.
- Or, events that are independent can become dependent, given additional information.

## Example: Cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
    - $A$ is the event that the card is a heart.
    - $B$ is the event that the card is a face card (J, Q, K).
- Are $A$ and $B$ independent?

## Example: Cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - $A$ is the event that the card is a heart.
  - $B$ is the event that the card is a face card (J, Q, K).
- Suppose you learn that the card is red. Are $A$ and $B$ independent given this new information?

## Conditional independence

- Recall that $A$ and $B$ are independent if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- $A$ and $B$ are **conditionally independent** given $C$ if:

$$\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

- Given that $C$ occurs, this says that $A$ and $B$ are independent of one another.

## Assuming conditional independence

- Sometimes we assume that events are conditionally independent to make calculations easier.
- Real-world events are almost never exactly conditionally independent, but they may be close.

## Example: Harry Potter and Discord

Suppose that 50% of UCSD students like Harry Potter and 80% of UCSD students use Discord. What is the probability that a random UCSD student likes Harry Potter and uses Discord, assuming that these events are conditionally independent given that a person is a UCSD student?

## Question 🤔

**Answer at q.dsc40a.com**

- Is it reasonable to assume conditional independence of:
    - liking Harry Potter
    - using Discord

  given that a person is a UCSD student?
- Is it reasonable to assume independence of these events in general, among all people?

**Which assumptions do you think are reasonable?**

A. Both.

B. Conditional independence only.

C. Independence (in general) only.

D. Neither.

## Independence vs. conditional independence

In general, **there is no relationship between independence and conditional independence**. All four scenarios before are possible:

1. $A$ and $B$ **are** independent, and **are** conditionally independent given $C$.
2. $A$ and $B$ **are** independent, but are **not** conditionally independent given $C$.
3. $A$ and $B$ are **not** independent, but **are** conditionally independent given $C$.
4. $A$ and $B$ are **not** independent, and are **not** conditionally independent given $C$.

## Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

**Scenario 1**: $A$ and $B$ **are** independent, and **are** conditionally independent given $C$.

## Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

**Scenario 2**: $A$ and $B$ **are** independent, but are **not** conditionally independent given $C$.

## Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

**Scenario 3**: $A$ and $B$ are **not** independent, but **are** conditionally independent given $C$.

## Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

**Scenario 4**: $A$ and $B$ are **not** independent, and are **not** conditionally independent given $C$.

# Summary

## Summary

- Two events $A$ and $B$ are **independent** when knowledge of one event does not change the probability of the other event.
  - Equivalent conditions: $\mathbb{P}(B|A) = \mathbb{P}(B)$, $\mathbb{P}(A|B) = \mathbb{P}(A)$, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.
- Two events $A$ and $B$ are **conditionally independent** given a third event, $C$, if they are independent given knowledge of event $C$.
  - Condition: $\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$.
- In general, there is no relationship between independence and conditional independence.
- **Next time**: Using Bayes' Theorem and conditional independence to solve the **classification problem** in machine learning.