Lecture 16

Independence and Conditional Independence

DSC 40A, Spring 2024

Announcements

- There is no live lecture today (Tuesday). Instead, the lecture video will be pre-recorded and posted on the course website by Tuesday morning.
    - There's also a lecture note I wrote for this lecture that you should read.
- Homework 7 is due on **Thursday at 11:59PM**.
- The final exam is soon: start practicing at practice.dsc40a.com!
    - There are tons of past probability exams, searchable by topic.

Agenda

- Independence.
- Conditional independence.

Remember, we've posted **many** probability resources on the resources tab of the course website. These will come in handy! Specific resources you should look at:

- The DSC 40A probability roadmap, written by Janine Tiefenbruck.
- The textbook Theory Meets Data, which explains many of the same ideas and contains more practice problems.

For combinatorics specifically, there are two supplementary videos I created that you should watch. Both are linked in this playlist, which is also linked at dsc40a.com.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

Independence

Updating probabilities

- Bayes' Theorem describes how to update the probability of one event, given that another event has occurred.

$$\underbrace{\mathbb{P}(B|A)}_{\text{new}} = \frac{\overbrace{\mathbb{P}(B)}^{\text{old}} \cdot \overbrace{\mathbb{P}(A|B)}^{\text{ratio}}}{\mathbb{P}(A)}$$

  - $\mathbb{P}(B)$ can be thought of as the "prior" probability of B occurring, before knowing anything about A.
  - $\mathbb{P}(B|A)$ is sometimes called the "posterior" probability of B occurring, given that A occurred.

- What if knowing that A occurred doesn't change the probability that B occurs? In other words, what if:

$$\overset{\text{new}}{\mathbb{P}(B|A)} = \overset{\text{old}}{\mathbb{P}(B)}$$

Independent events

- A and B are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$\boxed{\mathbb{P}(B|A) = \mathbb{P}(B)} \qquad \mathbb{P}(A|B) = \mathbb{P}(A)$$

equivalent statements

- Otherwise, A and B are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

with extra knowledge of A, $\mathbb{P}(B)$ is unchanged

→ Suppose $\mathbb{P}(B|A) = \mathbb{P}(B)$. Let's show $\mathbb{P}(A|B) = \mathbb{P}(A)$.

Bayes: $\mathbb{P}(B|A) = \dfrac{\mathbb{P}(B)\,\mathbb{P}(A|B)}{\mathbb{P}(A)} = \mathbb{P}(B) \Rightarrow \cancel{\mathbb{P}(B)}\,\mathbb{P}(A|B) = \cancel{\mathbb{P}(B)}\,\mathbb{P}(A)$

$\Rightarrow \mathbb{P}(A|B) = \mathbb{P}(A)$

Independent events

- **Equivalent definition**: A and B are independent events if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- To check if A and B are independent, use whichever is easiest:
  - $\mathbb{P}(B|A) = \mathbb{P}(B)$.
  - $\mathbb{P}(A|B) = \mathbb{P}(A)$.
  - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

General multiplication rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$$

<u>Only</u> when A, B independent:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Question 🤔

Answer at q.dsc40a.com

mutually exclusive: no overlap (can't happen at the same time)

Mutual exclusivity and independence

Suppose A and B are two events with non-zero probabilities. Is it possible for A and B to be both mutually exclusive and independent?

- A. Yes.
- B. No.

If A, B independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

If A, B mutually exclusive, $\mathbb{P}(A \cap B) = 0$

If both: $\mathbb{P}(A) \cdot \mathbb{P}(B) = 0$

$\Rightarrow \mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$

$\Rightarrow$ but we were told A, B have non-zero probabilities!

Example: Venn diagrams

For simplicity, let

$a = \mathbb{P}(A)$

$b = \mathbb{P}(B)$

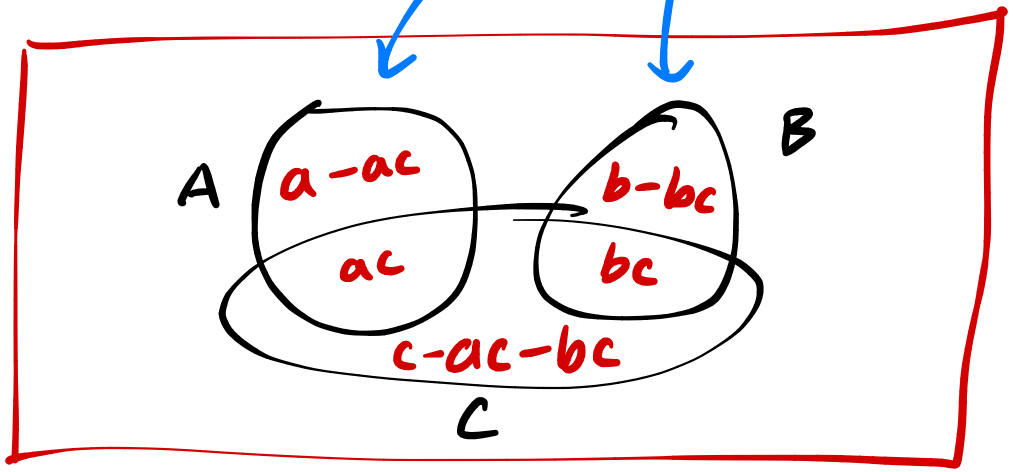
$c = \mathbb{P}(C)$

For three events A, B, and C, we know that:

- A and C are independent,
- B and C are independent,
- A and B are mutually exclusive,
- $\mathbb{P}(A \cup C) = \frac{2}{3}, \mathbb{P}(B \cup C) = \frac{3}{4}, \mathbb{P}(A \cup B \cup C) = \frac{11}{12}$.

Find $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(C)$.

A and B don't overlap!



Tip: A Venn Diagram is not necessary, but (potentially) helpful.

3 equations, 3 unknowns (a, b, c):

$\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C) = ac$
since A, C independent

① $\mathbb{P}(A \cup C) = a + c - ac = \frac{2}{3}$

② $\mathbb{P}(B \cup C) = b + c - bc = \frac{3}{4}$

③ $\mathbb{P}(A \cup B \cup C) = a + b + c - ac - bc = \frac{11}{12}$

$\Rightarrow$ ① + ② : $a + b + 2c - ac - bc = \frac{2}{3} + \frac{3}{4}$

③ : $a + b + c - ac - bc = \frac{11}{12}$

① + ② − ③ : $c = \frac{2}{3} + \frac{3}{4} - \frac{11}{12} = \frac{8}{12} + \frac{9}{12} - \frac{11}{12} = \frac{6}{12} = \boxed{\frac{1}{2}}$

c into ① : $a + \frac{1}{2} - \frac{1}{2}a = \frac{2}{3} \Rightarrow \frac{1}{2}a = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

$\Rightarrow a = 2 \cdot \frac{1}{6} = \boxed{\frac{1}{3}}$

c into ② : $b + \frac{1}{2} - \frac{1}{2}b = \frac{3}{4} \Rightarrow \frac{1}{2}b = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

$\Rightarrow b = 2 \cdot \frac{1}{4} = \boxed{\frac{1}{2}}$

$$\boxed{\mathbb{P}(A) = \frac{1}{3}, \quad \mathbb{P}(B) = \frac{1}{2}, \quad \mathbb{P}(C) = \frac{1}{2}}$$

Example: Cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - A is the event that the first card is a heart.
  - B is the event that the second card is a club.
- If you draw the cards **with** replacement, are A and B independent? Yes! $\mathbb{P}(B|A) = \frac{13}{52} = \mathbb{P}(B)$
- If you draw the cards **without** replacement, are A and B independent? $\boxed{\text{No!}}$

$\Rightarrow$ No: Once you remove one Heart, the remaining cards are less likely to be Hearts, and so more likely to be other suits, like clubs. $\mathbb{P}(B|A) = \frac{13}{51} \neq \frac{13}{52} = \mathbb{P}(B)$ if without replacement

Example: Cards

A   B

♥	: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦	: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣	: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠	: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw one card from a deck of 52.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Are A and B independent? $\boxed{\text{Yes!}}$

$\mathbb{P}(A) = \frac{13}{52} = \frac{1}{4}$, $\mathbb{P}(B) = \frac{12}{52} = \frac{3}{13}$,

$\mathbb{P}(A \cap B) = \frac{3}{52} = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Another interpretation of independence:

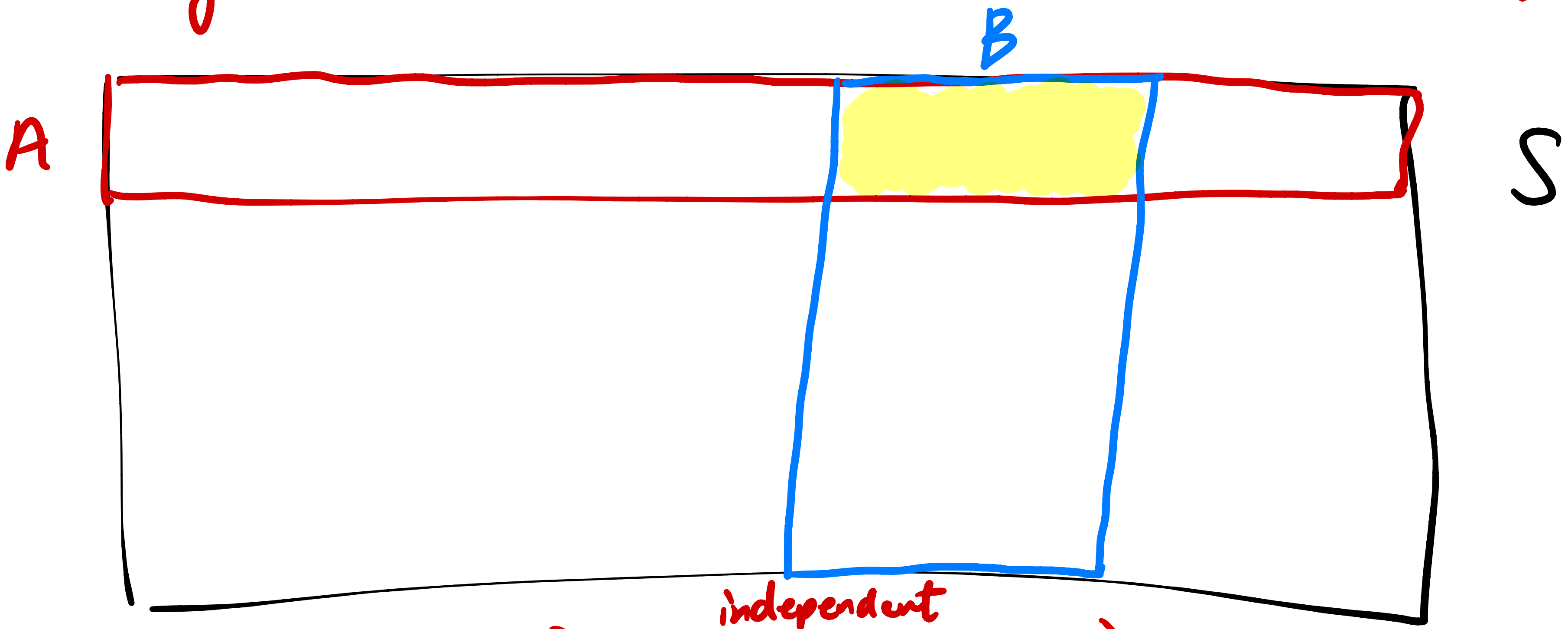
the proportion of face cards within A,

$\mathbb{P}(B \mid A) = \frac{3}{13}$

equals the proportion of face cards within the whole deck,

$\mathbb{P}(B) = \frac{12}{52} = \frac{3}{13}$

Visualizing independence when outcomes are equally likely:



sample space

$$\mathbb{P}(B|A) \overset{\text{independent}}{=\!=} \mathbb{P}(B)$$

the proportion of A taken up by B

the proportion of S taken up by B

Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

Example: Breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

$$\mathbb{P}(\text{Avo toast} \mid \text{DSC}) = \mathbb{P}(\text{Avo toast}) = \boxed{25\%}$$

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

$$\mathbb{P}(\text{Avo toast} \cap \text{DSC}) = \mathbb{P}(\text{Avo toast}) \cdot \mathbb{P}(\text{DSC})$$
$$= 0.25 \cdot 0.01 = 0.025 = \boxed{0.25\%}$$

Conditional independence

Conditional independence

- Sometimes, events that are dependent **become** independent, upon learning some new information.
- Or, events that are independent can become dependent, given additional information.

Example: Cards

A   B

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

King of Clubs gone! (bad dog)

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Are A and B independent? No!

$\mathbb{P}(B|A) = \frac{3}{13}$

$\mathbb{P}(B) = \frac{11}{51}$

not the same!

Another interpretation:

$\mathbb{P}(A) = \frac{13}{51}$, $\mathbb{P}(B) = \frac{11}{51}$,

$\mathbb{P}(A \cap B) = \frac{3}{51} \neq \frac{13}{51} \cdot \frac{11}{51} = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Example: Cards

A B

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

[redacted]

If we're given that our card is red, we must ignore all black cards!

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Suppose you learn that the card is red. Are A and B independent given this new information? Yes!

Within the red cards (i.e. given C):

$$P(B|A) = \frac{3}{13} = \frac{6}{26} = P(B)$$

⇒ Within the red cards, A and B are independent!

Conditional independence

- Recall that A and B are independent if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- A and B are **conditionally independent** given C if:

$$\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

comes from the definition of regular independence, but with "given C" added to all three terms

- Given that C occurs, this says that A and B are independent of one another.

Practically, one way to check:

$$\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \cdot \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}$$

Assuming conditional independence

- Sometimes we assume that events are conditionally independent to make calculations easier.
- Real-world events are almost never exactly conditionally independent, but they may be close.

Example: Harry Potter and Discord

Suppose that 50% of UCSD students like Harry Potter and 80% of UCSD students use Discord. What is the probability that a random UCSD student likes Harry Potter and uses Discord, assuming that these events are conditionally independent given that a person is a UCSD student?

$$\mathbb{P}\left((\text{like HP} \cap \text{use Discord}) \mid \text{UCSD}\right) = \mathbb{P}(\text{like HP} \mid \text{UCSD}) \cdot \mathbb{P}(\text{use Discord} \mid \text{UCSD})$$

$$= 0.5 \cdot 0.8$$

$$= \boxed{0.4}$$

Question 🤔

Answer at q.dsc40a.com

- Is it reasonable to assume conditional independence of:
    - liking Harry Potter
    - using Discord

    given that a person is a UCSD student?

Yes!

- Is it reasonable to assume independence of these events in general, among all people?

No!

Age

Which assumptions do you think are reasonable?

A. Both.

B. Conditional independence only.

C. Independence (in general) only.

D. Neither.

Independence vs. conditional independence

In general, **there is no relationship between independence and conditional independence**. All four scenarios before are possible:

1. A and B **are** independent, and **are** conditionally independent given C.
2. A and B **are** independent, but are **not** conditionally independent given C.
3. A and B are **not** independent, but **are** conditionally independent given C.
4. A and B are **not** independent, and are **not** conditionally independent given C.

Note: Slide order is slightly different than podcast!

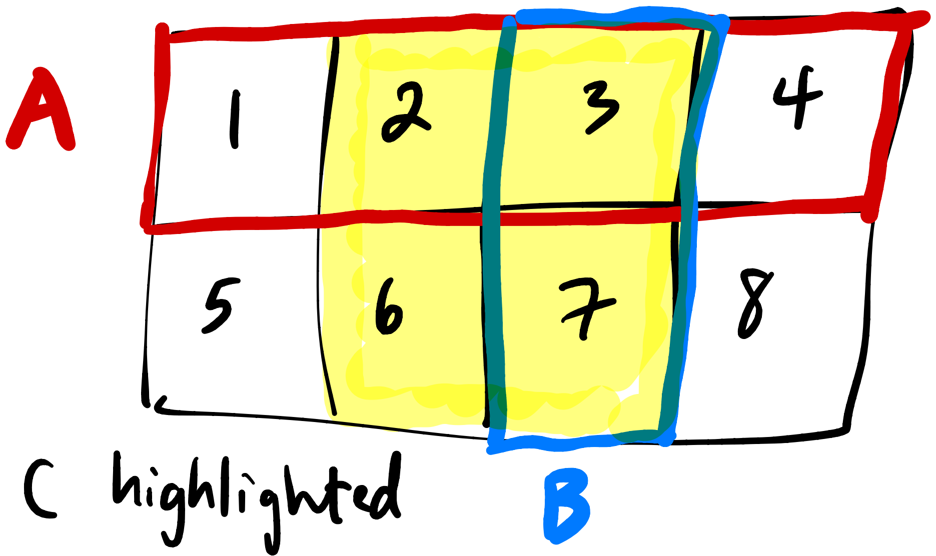
Exam-style question!

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

$A = \{1, 2, 3, 4\}$ $B = \{3, 7\}$ $C = \{2, 3, 6, 7\}$

Scenario 1: A and B **are** independent, and **are** conditionally independent given C.



A, B ind: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

$\frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4}$

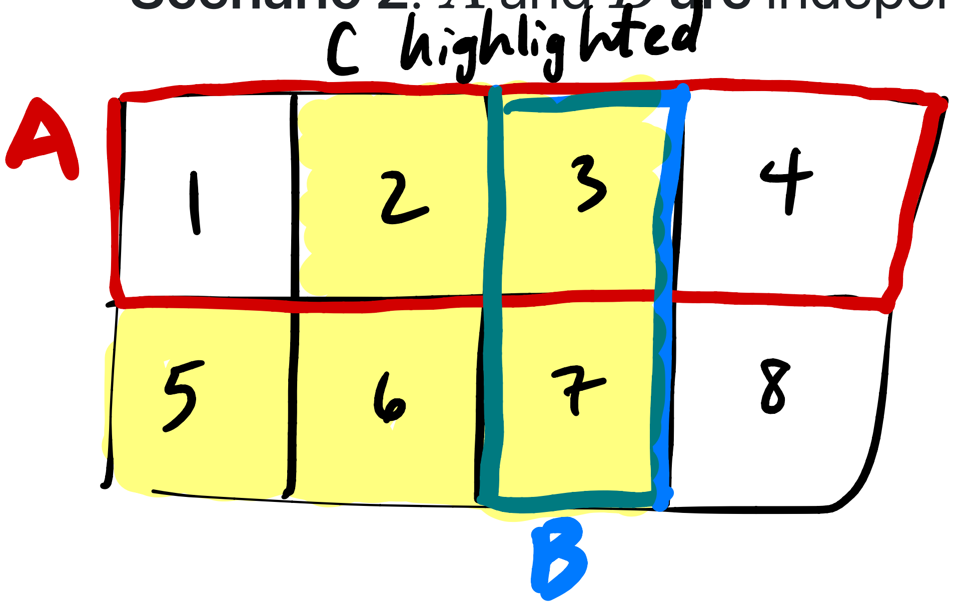
A, B cond. ind. given C: $\mathbb{P}((A \cap B) | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$

$\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$ ✓

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

Scenario 2: A and B are independent, but are **not** conditionally independent given C.



A, B ind: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

$\frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4}$

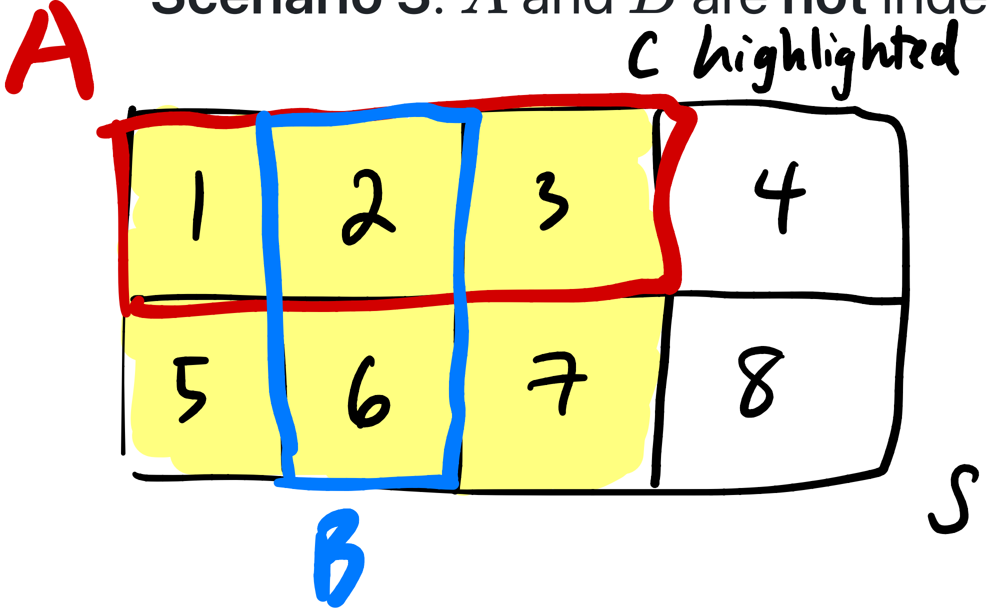
A, B <u>not</u> cond. ind. given C: $\mathbb{P}((A \cap B | C) \neq \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$

$\frac{1}{5} \neq \frac{2}{5} \cdot \frac{2}{5}$

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

Scenario 3: A and B are **not** independent, but **are** conditionally independent given C.



A, B not ind! $\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$

$$\frac{1}{8} \neq \frac{3}{8} \cdot \frac{1}{4}$$

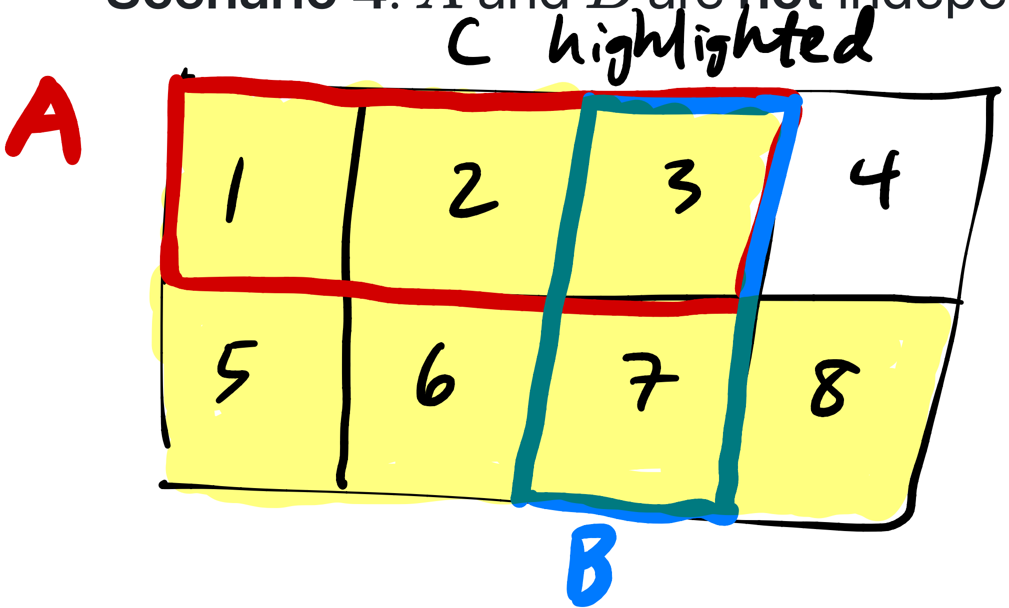
A, B cond. ind. given C: $\mathbb{P}((A \cap B) | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$

$$\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$$

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

Scenario 4: A and B are **not** independent, and are **not** conditionally independent given C.



A, B not ind! $\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$

$$\frac{1}{8} \neq \frac{3}{8} \cdot \frac{1}{4}$$

A, B not cond ind given C: $\mathbb{P}((A \cap B)|C) \neq \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$

$$\frac{1}{7} \neq \frac{3}{7} \cdot \frac{2}{7}$$

Summary

Summary

- Two events A and B are **independent** when knowledge of one event does not change the probability of the other event.
  - Equivalent conditions: $\mathbb{P}(B|A) = \mathbb{P}(B)$, $\mathbb{P}(A|B) = \mathbb{P}(A)$, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.
- Two events A and B are **conditionally independent** given a third event, C, if they are independent given knowledge of event C.
  - Condition: $\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$.
- In general, there is no relationship between independence and conditional independence.
- **Next time**: Using Bayes' Theorem and conditional independence to solve the **classification problem** in machine learning.