

Lecture 13

Combinatorics

DSC 40A, Summer 2024

Announcements

- Homework 5 is due **tonight**.

Agenda

- Recap: The domino problem.
- Combinatorics.
- Lots and lots of examples.

Remember, we've posted **many** probability resources on the [resources tab of the course website](#). These will come in handy! Specific resources you should look at:

- The [DSC 40A probability roadmap](#), written by Janine Tiefenbruck.
- The textbook [Theory Meets Data](#), which explains many of the same ideas and contains more practice problems.

For today's lectures specifically, there are two supplementary videos that you should watch. Both are linked in [this playlist](#), which is also linked at [dsc40a.com](#).

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions"
link in the top right corner of dsc40a.com.

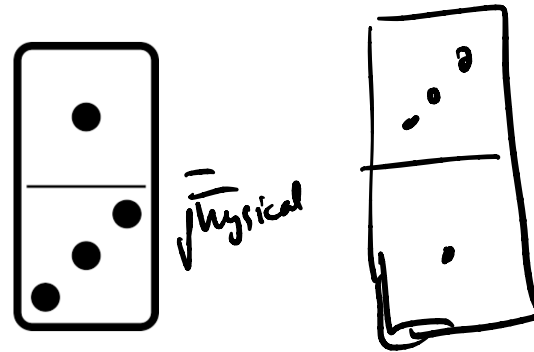
Recap: The domino problem

Example: Dominoes

(source: 538)

In a set of dominoes, each tile has two sides with a number of dots on each side: 0, 1, 2, 3, 4, 5, or 6. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.

7							
(0,0)	6						
(0,1)	(1,1)	5					
⋮	(1,2)	4					
	⋮	(2,2)	3				
		(2,3)	(3,3)	2			
		⋮	⋮	(4,4)	1		
				(4,5)	(5,5)		
(0,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	

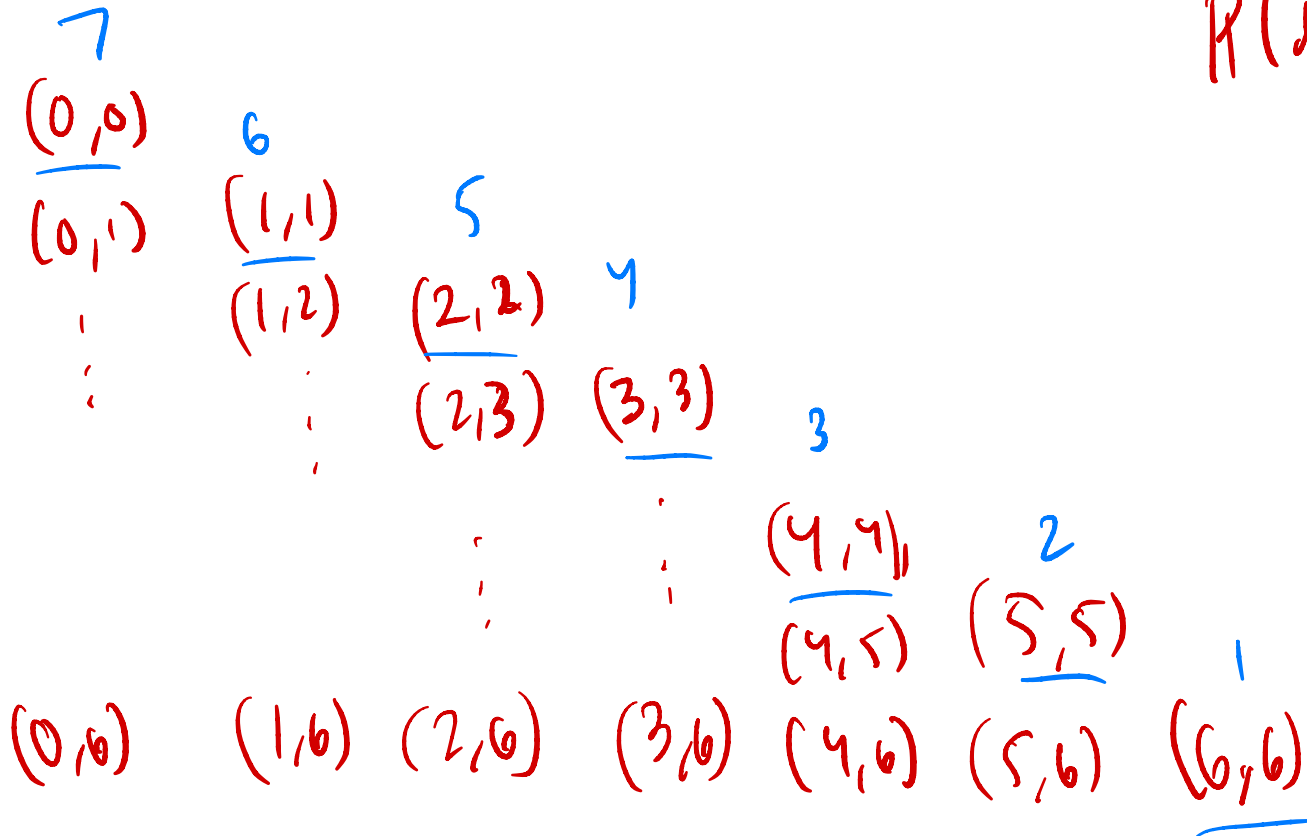


$$\begin{aligned} &1 + 2 + \dots + 7 \\ &= \frac{7(7+1)}{2} \\ &= 28 \end{aligned}$$

Example: Dominoes

Question 1: What is the probability of drawing a "double" from a set of dominoes – that is, a tile with the same number on both sides?

$$P(\text{double}) = \frac{\# \text{ doubles}}{\# \text{ tiles}} = \frac{7}{28} = \boxed{\frac{1}{4}}$$



Example: Dominoes

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

"Conditional probability"

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?

7

(0,0)

(0,1)

⋮

6

(1,1)

(1,2)

⋮

5

(2,2)

(2,3)

⋮

4

(3,3)

⋮

3

(4,4)

(4,5)

⋮

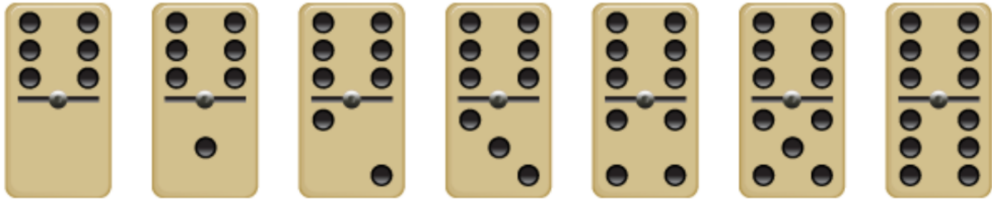
2

(5,5)

⋮

1

(6,6)



→ given

$P(\text{double} | \text{at least one 6}) =$

$= \frac{P(\text{double} \cap \text{at least one 6})}{P(\text{at least one 6})}$

$= \frac{\# \text{ doubles w/ at least one 6}}{\# \text{ at least one 6}}$

$= \frac{1}{7}$

$\frac{1/28}{7/28}$

if $P(B|A) = P(B)$,

A & B independent

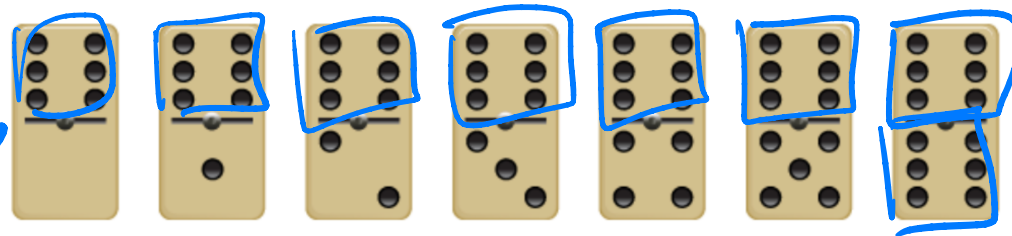
Example: Dominoes

$P(\text{double})$ from before

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has 6 dots. What is the probability that this tile is a double, with 6 on both sides?

28 dominoes
 $\Rightarrow 28 \times 2$ half dominoes

given we
picked one of



$$P(\text{double} \mid \text{uncovered a 6}) = \frac{2}{8} = \frac{1}{4}$$

$\rightarrow$ of the 8 half-dominoes equal to 6,
how many are across from another 6?

$$\frac{\frac{2}{8}}{\frac{8}{56}} = \frac{P(\text{double} \cap \text{uncovered 6})}{P(\text{uncovered a 6})}$$

2

See 538's explanation here.

Don't believe me? Believe the simulation!

- To verify your answer to a probability problem, you can often run a simulation!
- [This notebook](#) has a simulation of the domino problem.

Probability of a double:

```
In [46]: 1 is_double / n
```

```
Out[46]: 0.250091
```

Probability of double 6s, given that at least one side is a 6:

```
In [47]: 1 double_6 / at_least_one_6
```

```
Out[47]: 0.14351561030844512
```

Probability of double 6s, given that we uncovered a single side and it was a 6:

```
In [48]: 1 double_6 / one_side_6
```

```
Out[48]: 0.25142644852869017
```

Combinatorics

Motivation

- Many problems in probability involve counting.
 - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
 - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called combinatorics.

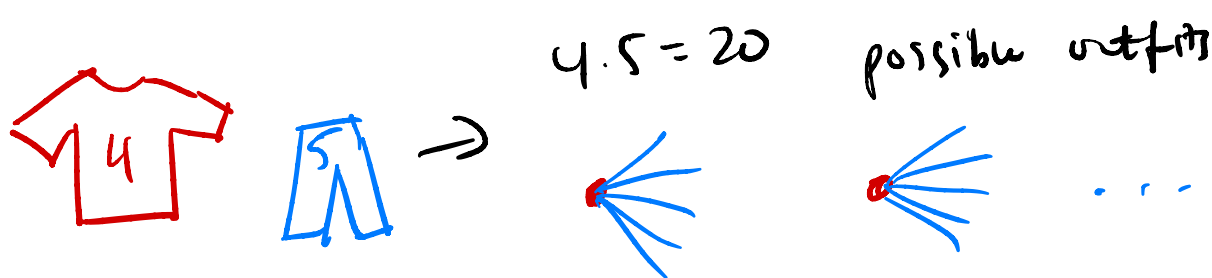
of ways to do something

selecting k items
from n possibilities

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
 - If drawing cards from a deck, the population is the deck of all cards.
 - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
 - Do we select elements with or without **replacement**?
 - Does the **order** in which things are selected matter?

Sequences



- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement** (i.e. **repetition is allowed**), such that **order matters**. $A 12345678 \neq A 87654321$: these are different
- Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times. How many such sequences are there?

$n = 52$
 $k = 4$

$\frac{52}{1^{st} \text{ card}} \quad \frac{52}{2^{nd}} \quad \frac{52}{3^{rd}} \quad \frac{52}{4^{th}} \rightarrow 52^4 \text{ sequences (or hands)}$

- Example:** A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

$n = 10$
 $k = 8$

$A \quad \frac{10}{1^{st}} \quad \frac{10}{2^{nd}} \quad \frac{10}{3} \quad \frac{10}{7} \quad \frac{10}{5} \quad \frac{10}{6} \quad \frac{10}{7} \quad \frac{10}{8} \rightarrow \underbrace{10 \times 10 \times \dots \times 10}_{8 \text{ times}} = 10^8$

Sequences

In general, the number of ways to select k elements from a group of n possible elements **with replacement (i.e. repetition is allowed)** and **order matters** is n^k .

$$\underbrace{n}_{1^{\text{st}} \text{ elem}} \cdot \underbrace{n}_{2^{\text{nd}}} \cdots \underbrace{n}_{(k-1)^{\text{th}}} \underbrace{n}_{k^{\text{th}}} = \underbrace{n \cdot n \cdots n}_{k \text{ times}} = n^k$$

don't jump straight to formula

$\Rightarrow$ write out the blanks

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement (i.e. repetition is not allowed), such that order matters.
- Example:** Draw 4 cards, without replacement, from a standard 52-card deck. How many such permutations are there? *can be anything except for first card*

$$n = 52$$
$$k = 4$$

$$\underbrace{52}_{1^{\text{st}} \text{ card}} \underbrace{51}_{2^{\text{nd}}} \underbrace{50}_{3^{\text{rd}}} \underbrace{49}_{4^{\text{th}}} = 52 \cdot 51 \cdot 50 \cdot 49$$

- Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$$n = 8$$
$$k = 3$$

$$\frac{8}{p} \frac{7}{vp} \frac{6}{s} = 8 \cdot 7 \cdot 6 = 336$$

$$\frac{8}{s} \frac{7}{p} \frac{6}{vp} = 8 \cdot 7 \cdot 6 = 336$$

Ex: $n=52$
 $k=4$

$n-k+1 = 52-4+1 = 52-3 = 49$

$P(52, 4) = 52 \cdot 51 \cdot 50 \cdot 49$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements **without replacement (i.e. repetition is not allowed)** and **order matters** is:

Permutations $\rightarrow P(n, k) = (n)(n-1)\dots(n-k+1)$

- To simplify: recall that the definition of $n!$ is:

$0! = 1$

$n! = n(n-1)(n-2)\dots(3)(2)(1)$

- Given this, we can write:

Important!

$$P(n, k) = \frac{n!}{(n-k)!} = \frac{n(n-1)\dots(n-k+2)(n-k+1)\cancel{(n-k)!}}{\cancel{(n-k)!}} = n(n-1)\dots(n-k+2)(n-k+1)$$

Aside

$6!$
 $= 6 \cdot 5!$
 $= 6 \cdot 5 \cdot 4!$
 $= 6 \cdot 5 \cdot 4 \cdot 3!$

Question 🤔

Answer at q.dsc40a.com

UCSD has 8 colleges. In how many ways can I rank my top 3 choices?

- A. 24. $8 \cdot 3$
- B. 336. $8 \cdot 7 \cdot 6$
- C. 512. 8^3
- D. 6561. 3^8
- E. None of the above.

$n=8$
 $k=3$

$$\underbrace{8}_{1^{st}} \cdot \underbrace{7}_{2^{nd}} \cdot \underbrace{6}_{3^{rd}} = \frac{8!}{(8-3)!} = \frac{8!}{5!}$$

don't try to
compute $8!$,
and $5!$

$$\frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}}$$

Special case of permutations $\rightarrow k=n$

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is:

$$\underbrace{n}_{1^{\text{st}}} \underbrace{n-1}_{2^{\text{nd}}} \underbrace{n-2}_{3^{\text{rd}}} \dots \underbrace{2}_{(n-1)^{\text{th}}} \underbrace{1}_{n^{\text{th}}} = (n)(n-1)\dots(2)(1) = n!$$

- This is consistent with the formula:

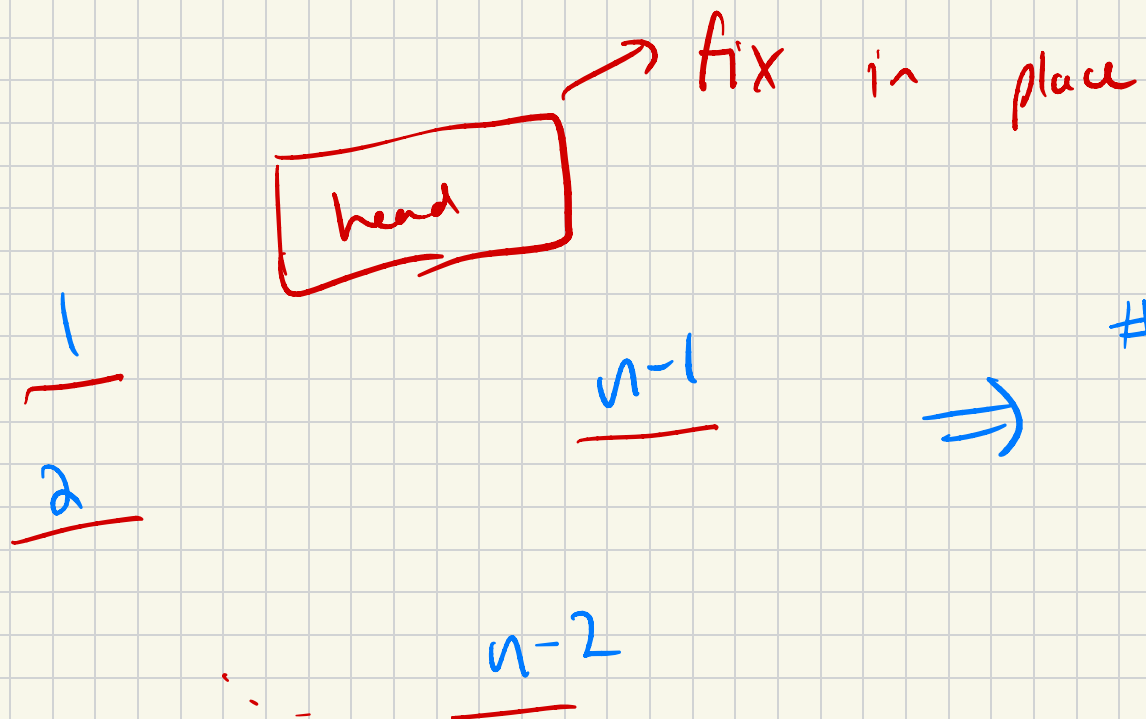
$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

- Followup:** How many ways are there to arrange n people in a circle?

ex $n=4$

$$\begin{array}{c} A \\ D \quad B \\ C \end{array} = \begin{array}{c} D \\ A \\ B \end{array} = \begin{array}{c} C \\ A \\ B \end{array} = \begin{array}{c} B \\ A \\ D \end{array} = \begin{array}{c} A \\ B \\ D \end{array} = \begin{array}{c} B \\ C \\ D \end{array} \Rightarrow \begin{array}{c} \# \text{ arrangements} \\ \Rightarrow \frac{4!}{4} = 3! \end{array}$$

$\rightarrow \text{general: } (n-1)!$



$\Rightarrow$ # arrangements

$$(n-1)(n-2)\dots(2)(1)$$

$$= (n-1)!$$

$$= \frac{n!}{n}$$

n rotations

Combinations

- A **combination** is a set of k elements selected from a group of n possible elements without replacement (i.e. repetition is not allowed), such that **order does not matter**.
- **Example:** There are 4 ice cream flavors. In how many ways can you pick two different flavors?

C V S T
chocolate vanilla strawberry Thai Tea

CV VS ST
CS VT
CT

⇒ 6 combinations

11
π

Not the answer

$$\frac{4}{1^{st}} \quad \frac{3}{2^{nd}} = \boxed{12}$$

Wrong! double counting

Generalize

Select 3 flavors from a set of 4

$$\underbrace{4}_{1^{\text{st}}} \quad \underbrace{3}_{2^{\text{nd}}} \quad \underbrace{2}_{3^{\text{rd}}} = \frac{24}{\text{permutations}}$$

adjustment $\rightarrow$

$$3! = \frac{24}{6} = \boxed{4 \text{ combinations}}$$

permutations / combination

Consider the Combination VCT. There are _____ ways of rearranging it that are all the same combination.

VCT VTC CVT CTV TCV TVC

$6 = 3! \rightarrow$ adjust by this for repeated permutations

Solution: divide 24 by $3!$

without replacement

From permutations to combinations

- There is a close connection between:

- the number of **permutations** of k elements selected from a group of n , and
- the number of **combinations** of k elements selected from a group of n .

order matters

order doesn't matter

$$\# \text{ combinations} = \frac{\# \text{ permutations}}{\# \text{ orderings of } k \text{ elements}}$$

adjustment of $k!$

- Since $\# \text{ permutations} = \frac{n!}{(n-k)!}$ and $\# \text{ orderings of } k \text{ elements} = k!$, we have:

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Combinations

In general, the number of ways to select k elements from a group of n elements **without replacement (i.e. repetition is not allowed) and **order does not matter** is:

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!k!}}$$

The symbol $\binom{n}{k}$ is pronounced " n choose k ", and is also known as the **binomial coefficient**.

$\binom{n}{k}$ not a fraction

↓
 $(a+b)^n = \dots$
math 183
elon 120a

permutation : $A \begin{smallmatrix} G \\ P \end{smallmatrix} \begin{smallmatrix} R \\ VP \end{smallmatrix} \Rightarrow \begin{smallmatrix} G \\ P \end{smallmatrix} \begin{smallmatrix} A \\ VP \end{smallmatrix} \begin{smallmatrix} R \\ S \end{smallmatrix}$

combination : $AGR \stackrel{\text{no roles}}{=} GAR$

Example: Committees

- How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$$\underbrace{8}_P \quad \underbrace{7}_{VP} \quad \underbrace{6}_S = 336 = 8 \cdot 7 \cdot 6$$

- How many ways are there to select a committee of 3 people from a group of 8 people?

$$\frac{336}{3!} = \binom{8}{3} = \frac{8!}{5!3!}$$

- If you're ever confused about the difference between permutations and combinations, come back to this example.
- More generally, don't jump straight to a formula: think about what the question is asking for.

Aside: Simplifying $\binom{n}{k}$

It's true that:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

However, when asked to simplify the value of $\binom{n}{k}$, do so strategically!

Ex

$$\begin{aligned}\binom{16}{3} &= \frac{16!}{13! 3!} = \frac{16 \cdot 15 \cdot 14 \cdot \cancel{13!}}{(3!) (\cancel{13!})} \\ &= \frac{\cancel{16}^5 \cdot \cancel{15}^3 \cdot 14}{\cancel{13} \cdot 2!} = 40 \cdot 14 = \boxed{560}\end{aligned}$$

Question 🤔

Answer at q.dsc40a.com

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in a set of dominoes? → 28

• A. $\binom{7}{2} \rightarrow \frac{7 \cdot 6 \cdot 5!}{2! \cdot 2!} = 21$

• B. $\binom{7}{1} + \binom{7}{2}$

• C. $P(7, 2)$

• D. $\frac{P(7,2)}{P(7,1)} 7!$

2 cases

double $\binom{7}{1}$

two unique halves $\binom{7}{2}$

$$\binom{n}{1} = n$$

$$\binom{n}{n-1} = n$$

$$\binom{7}{1} + \binom{7}{2} = 21 + 7$$

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	No, order doesn't matter
With replacement Repetition allowed	n^k possible sequences	more complicated: watch this video * <i>domino example</i>
Without replacement Repetition not allowed	$\frac{n!}{(n-k)!}$ possible permutations	$\binom{n}{k}$ combinations

**or see the previous slide.*

More examples

All we're going to do for the remainder of today's lecture and much of Wednesday's lecture is work through examples of combinatorics problems.

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $\mathbb{P}(A) = \frac{|A|}{|S|}$.  *# outcomes in event A*
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Overview: Selecting students

We're going to answer the same question using several different techniques.

There are 20 students in a class. ^{→ Avocado} Avi is one of them. Suppose we select 5 students in the class uniformly at random without replacement. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 1: Using permutations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$S = \{ \text{all permutations of 5 students selected from 20} \}$

denominator = $|S| = \frac{20!}{15!}$

$$\frac{20}{1^{st}} \cdot \frac{19}{2^{nd}} \cdot \frac{18}{3^{rd}} \cdot \frac{17}{4^{th}} \cdot \frac{16}{5^{th}}$$

$S = \{ \text{all permutations of 5 students selected from 20} \}$

denominator = $|S| = \frac{20!}{15!}$

$$\frac{20}{1^{\text{st}}} \frac{19}{2^{\text{nd}}} \frac{18}{3^{\text{rd}}} \frac{17}{4^{\text{th}}} \frac{16}{5^{\text{th}}}$$

Numerator: # permutations with Avi

5 cases

<u>A</u>	<u>19</u>	<u>18</u>	<u>17</u>	<u>16</u>
<u>19</u>	<u>A</u>	<u>18</u>	<u>17</u>	<u>16</u>
—	—	<u>A</u>	—	—
—	—	—	<u>A</u>	—
—	—	—	—	<u>A</u>

$\frac{19!}{15!} \cdot 5$
 "everyone else"
 cases

$$P = \frac{19 \cdot 18 \cdot 17 \cdot 16 \cdot 5}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}$$

$$= \frac{1}{4} = \frac{5}{20}$$

Selecting students

Method 2: Using permutations and the complement

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 3: Using combinations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 4: The "easy" way

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Question 🤔

Answer at q.dsc40a.com

With vs. without replacement

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

- A. Equal to.
- B. Greater than.
- C. Less than.

→ in video!

Even more examples

Watch the video linked below!

The following slides **will not** be covered in lecture. Instead, they're covered in [this walkthrough video](#).

1. 2 dogs and 2 cats?
2. 3 dogs and 1 cat?
3. At least 2 dogs?

① $\binom{5}{2} \cdot \binom{7}{2}$

② $\binom{5}{3} \binom{7}{1}$

③ at least 2 dogs
= 2 dogs, 2 cats OR
3 dogs, 1 cat OR
4 dogs, 0 cats

$\Rightarrow \binom{5}{2} \binom{7}{2} + \binom{5}{3} \binom{7}{1} + \binom{5}{4} \binom{7}{0}$

A screenshot from the walkthrough video.

We'll still cover more examples on Tuesday, but now you have a few more examples to refer to when working on Homework 6 this week.

Example: Pets

Part 1: We have 12 pets: 5 dogs and 7 cats. In how many ways can we select 4 pets?

Example: Pets

Part 2: We have 12 pets: 5 dogs and 7 cats. In how many ways can we select 4 pets such that we have...

1. 2 dogs and 2 cats?
2. 3 dogs and 1 cat?
3. At least 2 dogs?

Example: Pets

Part 3: We have 12 pets: 5 dogs and 7 cats. We randomly select 4 pets. What's the probability that we selected at least 2 dogs?

Example: A fair coin

Suppose we flip a fair coin 10 times.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Example: An unfair coin

Suppose we flip an **unfair** coin 10 times. The coin is biased such that for each flip, $\mathbb{P}(\text{heads}) = \frac{1}{3}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

