# DSC 40A - Group Work Session 6

due Monday, Feb 26 at 11:59pm

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **include all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive full credit, you must work in a group of two to four students for at least 50 minutes in your assigned discussion section. You can also self-organize a group and meet outside of discussion section for 80 percent credit.

## 1 Combinatorics

In probability, when all outcomes in the sample space are equally likely, the probability of an event is the number of outcomes in the event divided by the number of outcomes in the sample space. Thus, the probability of the event reduces to two counting (or combinatorics) questions, which ask *how many* outcomes are possible. When solving a counting question, it helps to write down one example outcome, then try to think about how many options we had at each step of generating this example.

There are a few basic combinatorial objects that we've studied in this class, namely sequences, permutations, and combinations. The hard part is often determining which one to use in which situation, which comes down to two important questions:

- Does the order in which I select the objects matter? In other words, does it count as different or the same if I choose the same objects in a different order?
- Am I selecting objects with or without replacement? In other words, am I allowed to have repeated objects?

**Sequences:**

A sequence is obtained by selecting $k$ elements from a group of $n$ possible elements with replacement, such that order matters. The number of sequences is

$$n^k.$$

**Permutations:**

A permutation is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order matters. The number of permutations is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

**Combinations:**

A combination is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order does not matter. The number of combinations is

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!}.$$

**Problem 1. Herb Garden**

You want to plant an herb garden, so you go to a garden store that has 50 different herbs: 28 are culinary herbs, 12 are medicinal herbs, and 10 are aromatic herbs. You select 5 herbs for your herb garden by taking a random sample **without replacement** from the 50 available herbs.

**a)** If you consider the herbs you select as a combination (i.e. the order in which you select each herb does not matter), how many combinations of 5 herbs are possible?

**Solution:** $C(50,5) = \binom{50}{5} = 2{,}118{,}760$

There are $C(50,5) = \binom{50}{5}$ possible sets of 5 herbs, chosen from 50.

**b)** If you consider the herbs you select as a combination (i.e. the order in which you select each herb does not matter), how many combinations of 5 herbs include 2 culinary herbs and 3 aromatic herbs?

**Solution:** $\binom{28}{2} \cdot \binom{10}{3} = 45{,}360$

There are $\binom{28}{2}$ ways to pick the culinary herbs and $\binom{10}{3}$ ways to pick the aromatic herbs, for a total of

$$\binom{28}{2} \cdot \binom{10}{3}$$

**c)** If you consider the herbs you select as a permutation (i.e. the order in which you select each herb matters), how many permutations of 5 herbs are possible?

**Solution:** $P(50,5) = 254{,}251{,}200$

There are $P(50,5) = 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46$ possible sequences of 5 distinct herbs from 50 options.

**d)** If you consider the herbs you select as a permutation (i.e. the order in which you select each herb matters), how many permutations of 5 herbs include 2 culinary herbs and 3 aromatic herbs?

**Solution:** $\binom{28}{2} \cdot \binom{10}{3} \cdot 5! = 28 \cdot 27 \cdot 10 \cdot 9 \cdot 8 \cdot \binom{5}{3} = 5{,}443{,}200$

There are $\binom{28}{2}$ ways of picking two culinary herbs and $\binom{10}{3}$ ways of picking three aromatic herbs, and so there are

$$\binom{28}{2} \cdot \binom{10}{3}$$

ways of picking 2 culinary herbs and three aromatic herbs (this is the answer to part (d)). But the order matters. For each combination of 5 herbs, there are 5! different orderings. The total number of sequences is therefore

$$\binom{28}{2} \cdot \binom{10}{3} \cdot 5!$$

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Here is another approach. We have to fill a sequence of 5 slots. To generate a sequence, we'll first pick the order in which the culinary herbs will appear. We'll then pick the order in which the aromatic herbs will appear. Then we will pick where the aromatic herbs will be placed in the order of five herbs.

There are $28 \cdot 27$ ways in which to order 2 culinary herbs.

There are $10 \cdot 9 \cdot 8$ ways in which to order 3 aromatic herbs.

There are $\binom{5}{3}$ ways of picking where to place the aromatic herbs in the final sequence.

The total number of sequences is

$$28 \cdot 27 \cdot 10 \cdot 9 \cdot 8 \cdot \binom{5}{3}$$

You can verify that this gives the same answer as above.

**e)** What is the probability that you choose 2 culinary herbs and 3 aromatic herbs for your garden?

**Solution:** $\frac{\binom{28}{2} \cdot \binom{10}{3}}{\binom{50}{5}} = \frac{\binom{28}{2} \cdot \binom{10}{3} \cdot 5!}{P(50,5)} = \frac{162}{7567} \approx 0.021$

We can solve this probability question using a sample space of permutations or combinations. To use combinations, divide the answer to part (b) by the answer to part (a):

$$\frac{\binom{28}{2} \cdot \binom{10}{3}}{\binom{50}{5}} = \frac{162}{7567} \approx 0.021.$$

To use permutations, divide the answer to part (d) by the answer to part (c):

$$\frac{\binom{28}{2} \cdot \binom{10}{3} \cdot 5!}{P(50,5)} = \frac{162}{7567} \approx 0.021$$

**Problem 2. Shuffling Strings**

**a)** How many different strings can be created by shuffling the letters of DOG?

**Solution:** There are 3 options for the first character, 2 for the second, and 1 for the third. This gives $3! = 6$.

**b)** How many different strings can be created by shuffling the letters of GAG?

*Hint:* The answer is not 6.

**Solution:** If you enumerate all of the possibilities, you'll see there are only 3 permutations of GAG: AGG, GAG, and GGA.

It's tempting to jump to the formula for permutations, which would imply that there are $3! = 6$ permutations of GAG. However, this double counts each permutation. Specifically, each unique permutation is counted once for every arrangement of the two Gs.

To see this concretely, suppose that instead our string is $G_1AG_2$. Then, there are 6 unique permutations:

$$G_1AG_2 \qquad G_2AG_1 \qquad AG_1G_2 \qquad AG_2G_1 \qquad G_1G_2A \qquad G_2G_1A$$

However, if $G_1$ and $G_2$ are both just treated as being Gs, then the first two permutations are the same, the second two permutations are the same, and the last two permutations are the same. So, we need to **divide** our result from using the permutation formula, 3!, by the number of arrangements of the repeated characters. In this case there are 2 repeated characters, so there are 2! ways to arrange them. Thus, the total number of permutations of GAG is $\frac{3!}{2!} = 3$.

Another way to think about this problem is that we have 3 characters and need to choose 2 of them to be Gs; we can do this in $\binom{3}{2}$ ways.

**c)** How many different strings can be created by shuffling the letters of GAAAGGGG?

*Hint:* How can you use combinations?

**Solution:**

The easiest way to think about this problem is using combinations.

There are 8 positions for characters to be placed in any given string, and we want to choose 3 of them to be A. This can be done in $\binom{8}{3} = 56$ ways. Equivalently, we could choose 5 of the positions to be G. This can be done in $\binom{8}{5} = 56$ ways, as well.

**d)** How many different strings can be created by shuffling the letters of AGGRAVATE?

**Solution:** AGGRAVATE has:

- 3 As,
- 2 Gs,
- 1 R,
- 1 V,
- 1 T, and
- 1 E.

One way to think about this problem is to start with 9!, which would be the total number of ways to rearrange the letters of AGGRAVATE if all letters were unique. Then, we need to account for the fact that there are repeated As and Gs, meaning that we overcounted. There are 3! ways to arrange the As in place and 2! ways to arrange the As in place, meaning that each unique permutation of AGGRAVATE was counted $3! \cdot 2!$ times instead of once.

Therefore, the actual number of permutations of AGGRAVATE is

$$\frac{9!}{3!2!} = 30240$$

Another approach uses combinations. Of the 9 positions, we need to select 3 of them to be filled with As. Then of the remaining positions, we need to select 2 of them to be filled with Gs. Then we need to select one of the remaining positions to contain the R, and so on, yielding

$$\binom{9}{3} * \binom{6}{2} * \binom{4}{1} * \binom{3}{1} * \binom{2}{1} * \binom{1}{1} = 30240$$

**Problem 3. Xs and Os**

Let $N(a, b)$ represent the number of strings you can create out of $a$ Xs and $b$ Os. Explain why $N(a, b)$ satisfies each of the following:

$$N(0, b) = 1 \quad (1)$$

$$N(a, 0) = 1 \quad (2)$$

$$N(a, b) = N(a-1, b) + N(a, b-1) \quad \text{for } a > 0 \text{ and } b > 0. \quad (3)$$

**Solution:** (1) If we have no Xs and $b$ Os, then we can create just one string (with all Os).

(2) If we have no Os and $a$ Xs, then we can create just one string (with all Xs).

(3) Every string that we create must start with an X or O. Notice that these cases are disjoint, as a string can start with only one letter. If a string with $a$ Xs and $b$ Os starts with an X, then the rest of the string has $a-1$ Xs and $b$ Os. So the number of such strings is $N(a-1,b)$. Similarly, rf a string with $a$ Xs and $b$ Os starts with an O, then the rest of the string has $a$ Xs and $b-1$ Os. So the number of such strings is $N(a,b-1)$. Since these two disjoint cases (starting with an X or starting with an O) cover all possibilities, we have $N(a,b) = N(a-1,b) + N(a,b-1)$.

### Problem 4. Tiebreaker

To break a tie among a group of $n \geq 3$ people, you come up with the following tiebreaker: Everyone flips a coin. If one person's coin is different from all the others, that person wins, and the tie is broken! Otherwise, repeat the process.

**a)** What is the probability that the tie is broken after the first coin toss?

**Solution:** $\frac{n}{2^{n-1}}$.

We can think of an outcome as a sequence of heads and tails, corresponding to each person's flip. Each coin toss sequence of length $n$ is equally likely, and there are $2^n$ possible coin toss sequences. The number of coin toss sequences with exactly one head is $n$, since there are $n$ possibilities for where to place the one head. Similarly, the number of coin toss sequences with exactly one tail is $n$. So the total number of coin toss sequences where one person is different from all the others is $n + n = 2n$. Therefore the probability of the tie being broken after the first coin toss is

$$\frac{2n}{2^n} = \frac{n}{2^{n-1}}.$$

**b)** Fix an integer $k \geq 1$. Find the probability that the tie is broken after exactly $k$ coin tosses?

**Solution:** $\left(1 - \frac{n}{2^{n-1}}\right)^{k-1} * \frac{n}{2^{n-1}}$.

If the tie is broken after exactly $k$ coin tosses, that means the first $k-1$ coin tosses did not have a winner, and the $k$th coin toss did. The probability of a single toss breaking the tie was found in part (a), so the probability of the $k$th toss breaking the tie is

$$\left(1 - \frac{n}{2^{n-1}}\right)^{k-1} * \frac{n}{2^{n-1}}.$$