DSC 40A - Group Work Session 7

due Monday, March 4 at 11:59pm

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **include all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive full credit, you must work in a group of two to four students for at least 50 minutes in your assigned discussion section. You can also self-organize a group and meet outside of discussion section for 80 percent credit.

1 Bayes' Theorem and the Law of Total Probability

Bayes' Theorem describes how to update the probability of one event given that another has occurred.

$$P(B|A) = \frac{P(A|B) * P(B)}{P(A)}$$

You can think of Bayes' Theorem as a restatement of the multiplication rule. If we multiply both sides of the expression above by $P(A)$, we get two equivalent expressions for $P(A \cap B)$ using the multiplication rule.

Another useful rule that is helpful in many Bayes' Theorem problems is the **Law of Total Probability** which says that if we have events $E_1, E_2, \ldots, E_k$ that partition our sample space,

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_k).$$

Problem 1.

There are two boxes. Box 1 contains three red and five white balls and box 2 contains two red and five white balls. A box is chosen at random, with each box equally likely to be chosen. Then, a ball is chosen at random from this box, with each ball equally likely to be chosen. The ball turns out to be red. What is the probability that it came from box 1?

Solution: $\frac{21}{37}$

We can find $P(\text{box 1}|\text{red})$ using Bayes' Theorem. The Law of Total Probability is useful to rewrite the denominator.

$$
\begin{aligned}
P(\text{box 1}|\text{red}) &= \frac{P(\text{red}|\text{box 1}) \cdot P(\text{box 1})}{P(\text{red})} \\
&= \frac{P(\text{red}|\text{box 1}) \cdot P(\text{box 1})}{P(\text{red}|\text{box 1}) \cdot P(\text{box 1}) + P(\text{red}|\text{box 2}) \cdot P(\text{box 2})} \\
&= \frac{(1/2)(3/8)}{(1/2)(3/8) + (1/2)(2/7)} \\
&= \frac{3/16}{3/16 + 1/7} \\
&= \frac{3/16}{(21+16)/(16 \cdot 7)} \\
&= \frac{3}{16} \cdot \frac{16 \cdot 7}{37} \\
&= \frac{21}{37}
\end{aligned}
$$

2 Independence and Conditional Independence

Recall that two events A, B are **independent** if knowledge of one event occurring does not affect the probability of the other event occurring. There are three equivalent definitions of independence:

$$P(A|B) = P(A) \tag{1}$$

$$P(B|A) = P(B) \tag{2}$$

$$P(A \cap B) = P(A) * P(B) \tag{3}$$

Two events that are not independent are also called **dependent**.

Two events A and B are **conditionally independent given** C if

$$P((A \cap B)|C) = P(A|C) * P(B|C).$$

Notice the similarity between this definition and the third definition of independence given above. Conditional independence given C means that *when C occurs*, A and B are independent in that case. But they may or may not be independent in general.

Problem 2.

Let A and B be events in a sample space with $0 < P(A) < 1$ and $0 < P(B) < 1$. If A is a subset of B, can A and B be independent? If yes, give an example, otherwise prove why not.

Solution: No, it is not possible for A and B to be independent.

If A is a subset of B, then $P(B|A) = 1$, but $P(B) < 1$, so $P(B|A) \neq P(B)$, which means A and B are not independent. If one event is a subset of the other, then the events are not independent, which makes sense because knowledge of one event tells you something about the chance of the other event occurring.

Problem 3.

Consider two flips of a fair coin. The sample space is $S =$ all outcomes of 2 flips of a coin $= \{HH, HT, TH, TT\}$, where each has equal probability $\frac{1}{4}$. We define the event A as $A =$ first flip is heads $= \{HH, HT\}$, and the event B as $B =$ second flip is heads $= \{HH, TH\}$. You can verify that A and B are independent by showing $P(A \cap B) = P(A) * P(B)$.

Now, suppose that the coin is not fair, and instead flips heads the first time with probability p and flips heads the second time with probability q.

Are A and B still independent?

Solution: Let's compute $P(A)$, $P(B)$, and $P(A \cap B)$ and see if $P(A \cap B) = P(A)P(B)$.

First, $P(A \cap B) = P(HH) = pq$.

Then,

$$P(A) = P(HH) + P(HT) = pq + p(1-q) = pq + p - pq = p,$$

and

$$P(B) = P(HH) + P(TH) = pq + (1-p)q = pq + q - pq = q.$$

This establishes that $P(A)P(B) = pq = P(A \cap B)$, so A and B are still independent.

Problem 4.

A box contains two coins: a regular coin and one fake two-headed coin ($P(H) = 1$). Choose a coin at random and flip it twice. Define the following events.

- A: First flip is heads (H).
- B: Second flip is heads (H).
- C: Coin 1 (regular) has been selected.

Are A and B independent? Are A and B conditionally independent given C?

Prove your answers using the definitions of independence and conditional independence. Also explain your answers intuitively.

Solution: Firstly, we know that $P(A|C) = P(B|C) = 1/2$ since a fair coin has equal probability of heads and tails. Also, given that Coin 1 is picked, we have $P((A \cap B)|C) = (1/2)(1/2) = 1/4$ (the probability of getting two heads in two tosses of a fair coin). Therefore, $P((A \cap B)|C) = P(A|C)P(B|C)$. This shows that **A and B are conditionally independent given C**.

Now, we will show that $P(A \cap B) \neq P(A)P(B)$.

To find $P(A), P(B)$ and $P(A \cap B)$, we use the law of total probability:

$$\begin{aligned} P(A) &= P(A|C)P(C) + P(A|\bar{C})P(\bar{C}) \\ &= (1/2) * (1/2) + 1 * (1/2) \\ P(A) &= 3/4 \end{aligned}$$

Similarly,

$$\begin{aligned} P(B) &= P(B|C)P(C) + P(B|\bar{C})P(\bar{C}) \\ &= (1/2) * (1/2) + 1 * (1/2) \\ P(B) &= 3/4 \end{aligned}$$

Finally,

$$
\begin{aligned}
P(A \cap B) &= P((A \cap B)|C)P(C) + P((A \cap B)|\bar{C})P(\bar{C}) \\
&= (1/4) * (1/2) + (1) * (1/2) \text{ since a two-headed coin means the first two flips are heads} \\
P(A \cap B) &= 5/8.
\end{aligned}
$$

Since $P(A \cap B) = 5/8 \neq P(A)P(B) = 9/16$, A **and** B **are not independent**. We can also justify this intuitively. For example, if we know that A has occurred (i.e., the first coin toss has resulted in heads), we would think that there is a higher chance of having chosen Coin 2 than Coin 1. This in turn increases the conditional probability of B occurring (in other words, $P(B|A) > P(B)$), which suggests that A and B are not independent. On the other hand, A and B are independent given C, because for a fair coin, the outcome of the second toss is not influenced by the result of the first toss. Generally speaking, conditional independence neither implies (nor is it implied by) independence.