Lecture 17 - Sequences, Permutations, and Combinations

DSC 40A, Winter 2024

Announcements

- Homework 5 is due **Tonight**.
    - Please come to office hours if you have questions!
- **Important:** We've posted **many** probability resources on the resources tab of the course website. These will no doubt come in handy.
    - No more DSC 40A-specific readings, though the Probability Roadmap was written specifically for students of this course.

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HDSI Undergraduate Social

All HDSI Undergrads and Faculty Welcome

Friday February 23rd
3-5pm
HDSI Patio

Join us for light refreshments and meet fellow students and faculty before finals start!

DDS AND HDSI DEI COMMITTEE JOINT EVENT

Agenda

- Sequences, permutations, and combinations.

Sequences, permutations, and combinations

Motivation

- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called **combinatorics**.

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times.

$n = 52$

$k = 4$

One possibility: $\diamond Q, \diamond Q, \diamond Q, \diamond Q$

How many possibility?

$$52 \cdot 52 \cdot 52 \cdot 52 = (52)^4 = n^k$$

- **Example:** A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

$n = 10$

$k = 8$

ex) A12345678

no choice = 1

10 options: 0-9

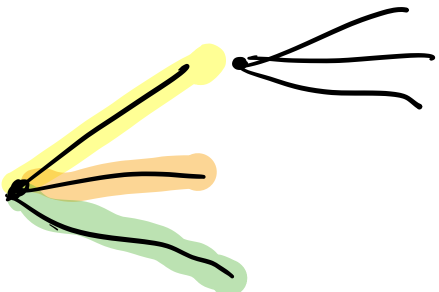
$$= n^k$$

$$1 \cdot (10 \cdot 10 \cdots 10) = 10^8$$

Sequences

In general, the number of ways to select *k* elements from a group of *n* possible elements such that **repetition is allowed** and **order matters** is n^k.

$k=3$ clustering



k^n

(Note: We mentioned this fact in the lecture on clustering!)

Permutations

- A **permutation** is obtained by selecting *k* elements from a group of *n* possible elements **without replacement**, such that **order matters**.

- **Example:** Draw 4 cards (without replacement) from a standard 52-card deck.

$n=52$

$k=4$

One possibility: ♢Q, ♤J, ♧10, ♡3

How many possibilities: $52 \cdot 51 \cdot 50 \cdot 49$ $52-4+1$ $n-k+1$

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

ex: give each person a name: a, b, c, d, e, f, g, h.

VP = a
P = f
Sec = d

$8 \cdot 7 \cdot 6$

$n-k+1$
$=8-3+1=6$

$n=8,\ k=3$

Permutations

$P(n,k) = \frac{n!}{(n-k)!}$

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$P(n,k) = (n)(n-1)\ldots(n-k+1)$$

- To simplify: recall that the definition of $n!$ is

$$n! = (n)(n-1)\ldots(2)(1)$$

- Given this, we can write

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$\frac{n!}{(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)\cdot\cancel{(n-k)}\cdot\cancel{(n-k-1)}\cdots\cancel{2}\cdot\cancel{1}}{\cancel{(n-k)}\cdot\cancel{(n-k-1)}\cdot\cancel{(n-k-2)}\cdots\cancel{2}\cdot\cancel{1}}$$

does the order matter? True

Can we have repetition? False

Permutation. $n=7$ $k=3$

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?

a) 21
b) 210
c) 343
d) 2187
e) None of the above.

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$P(7,3) = \frac{7\cdot 6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{\cancel{4\cdot 3\cdot 2\cdot 1}} = 7\cdot 6\cdot 5$$

ex) Seventh . Sixth . Muir

7 6 5

Special case of permutations

$n = n, \; k = n$

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is

$$\underbrace{n}_{1^{st}\text{ in line}} \cdot \underbrace{(n-1)}_{2^{nd}\text{ in line}} \cdot (n-2) \cdot (n-3) \cdots (2) \cdot (1) = n!$$

- This is consistent with the formula

$$P(n, \overset{k}{n}) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

$1! = 1$

$0! = \frac{1!}{1} = \frac{1}{1} = 1$

Combinations

order doesn't matter

- A **combination** is a set of k items selected from a group of n possible elements **without replacement**, such that **order does not matter**.

" False　　　　　　" False

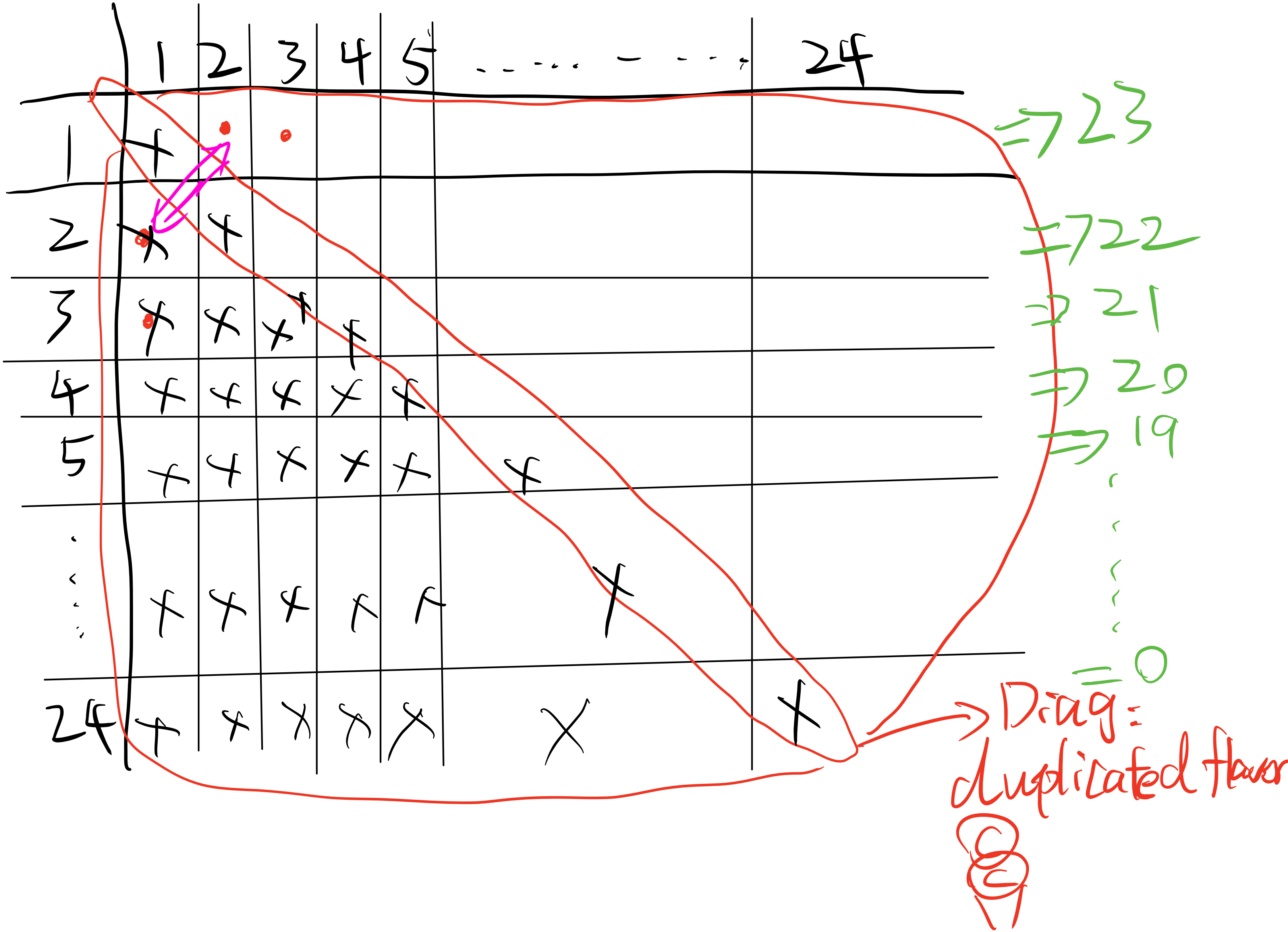
- **Example:** There are 24 ice cream flavors. How many ways can you pick two different flavors?

How many 2-flavor combination?

$P(24,2) = 24 \cdot 23$

$$\frac{24 \cdot 23}{2}$$

1. Count permutations where order = True
2. adjust the # of possibilities where order doesn't matter.


1 2 3 4 5 24
1
2
3
4
5
24
⇒ 23
⇒ 22
⇒ 21
⇒ 20
⇒ 19
= 0
Diag:
duplicated flavor

$23 + 22 + 21 \cdots\cdots\cdots\cdots 3 + 2 + 1$

24

24

24

How many pairs of 24 do I have?

$$\frac{23}{2} = 11.5$$

$$\text{total} = 11.5 \cdot 24 = \frac{23}{2} \cdot 24$$

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24 flavors in total, How many 3-flavor combos?

$$P(24,3) = \frac{24!}{21!} \qquad \frac{P(24,3)}{3!} = \frac{24!}{21! \cdot 3!}$$

How many possible ways can I re-arrange 3-flavor? $3!$

From permutations to combinations

- There is a close connection between:
  - the number of permutations of k elements selected from a group of n, and
  - the number of combinations of k elements selected from a group of n

$$\#\text{ combinations} = \frac{\#\text{ permutations}}{\#\text{ orderings of } k \text{ items}}$$

- Since # permutations = $\frac{n!}{(n-k)!}$ and # orderings of k items = $k!$, we have

n choose k.

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Combinations

In general, the number of ways to select k elements from a group of n elements such that **repetition is not allowed** and **order does not matter** is

False

False

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

The symbol $\binom{n}{k}$ is pronounced “n choose k”, and is also known as the **binomial coefficient**.

Example: committees

- How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$8 \cdot 7 \cdot 6$

order matter (permutation)

- How many ways are there to select a committee of 3 people from a group of 8 people?

$\frac{8 \cdot 7 \cdot 6}{3!} = 8 \cdot 7$

order doesn't matter (combination)

- If you're ever confused about the difference between permutations and combinations, **come back to this example**.

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in the set of dominoes?

a) $\binom{7}{2}$

b) $\binom{7}{1} + \binom{7}{2}$

c) $P(7, 2)$

d) $\frac{P(7,2)}{P(7,1)} 7!$

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.
- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.
- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.