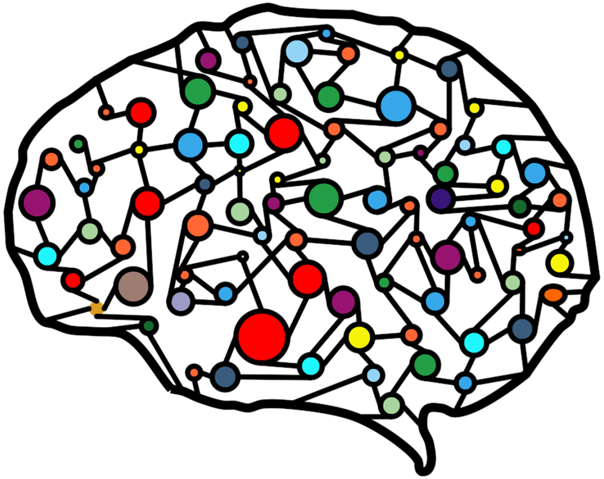
Lecture 19 - More Probability and Combinatorics Examples



DSC 40A, Winter 2024

Announcements

- Discussion is tonight.
- Homework 6 has been released, due **Wednesday at 11:59pm**.
- Homework 7 (last HW) will be released this Friday along with the second Extra Credit opportunity.

Agenda

- Lots of examples.

Selecting students — overview

We're going answer the same question using several different techniques.

All students are equally likely to be selected

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

False

① Solve this with Order → True.

P(n,k)

Give 'names' to all students:

A B C D ...... T = 20 students

Avi

Selecting students (Method 1: using permutations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = a permutation (ordered selection) of 5 students chosen from A, B ... T

ex) LPFGA, LFGAT ....

$$P(A \text{ included}) = \frac{\#\text{ permutation w/ A}}{\text{total \# of permutations}}$$

Probability

$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

$= P(20, 5)$

Permutation

Numerator: # of permutations including A.

ex). ACJTE

A _ _ _ _ $\rightarrow 1 \times 19 \times 18 \times 17 \times 16$

$1 \times 19 \times 18 \times 17 \times 16$

5 cases

A _ _ _ _

_ A _ _ _

_ _ A _ _

_ _ _ A _

_ _ _ _ A

$$\boxed{\frac{5 \cdot P(19,4)}{P(20,5)} = \frac{5 \cdot \frac{19!}{\cancel{15!}}}{\left(\frac{20!}{\cancel{15!}}\right)} = 5 \cdot \frac{19!}{20!} = \frac{5}{20} = 1/4}$$

total # of perm

is

$5 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

$= 5 \cdot P(19,4)$

Selecting students (Method 2: using permutations and the complement)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$\frac{\#\text{ Perms including A}}{\text{total \# Perms}} = \frac{\text{total \# of Perms} - \#\text{ Perms not include A}}{\text{total \# of Perms}}$$

$$\#\text{ perms not include A} = 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 = P(19,5)$$

$$\frac{P(20,5) - P(19,5)}{P(20,5)} = \frac{1}{4}$$

Selecting students (Method 3: using combinations)

Order = False

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = set of 5 students chosen from A, B, ..., T

Order = False.

ex) {B, D, G, H, M}

$$P(A \text{ included}) = \frac{\#\text{ of sets of 5 students including A}}{\#\text{ of sets of 5 students}}$$

Probability

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

\# sets of 5 students: $C(20,5) = \binom{20}{5}$

$= \frac{2!}{15!5!}$

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Avi?

sets include Avi

$n = 19$

$k = 4$

$C(n,k) = C(19,4)$

# of other students except A
B, C, ..., T

choose 4 other students to go w/ Avi

$P(\text{A included}) = \frac{C(19,4)}{C(20,5)} = \frac{1}{4}$

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$\frac{C(19,4)}{C(20,5)} = \frac{\frac{19!}{15!\,4!}}{\frac{20!}{15!\,5!}} = \frac{19!}{\cancel{15!}\,4!} * \frac{\cancel{15!}\,5!}{20!}$$

$$= \frac{19!}{20!} \cdot \frac{5!}{4!} = \frac{1}{20} \cdot 5 = \boxed{\frac{1}{4}}$$

Selecting students (Method 4: “the easy way”)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

① randomize all 20 students → "good" position
② line them up in random order
③ choose the first 5 students

S = positions where A may end up.

$$\frac{\#\text{ "good" position}}{\text{Total }\#\text{ of positions}} = \frac{5}{20} = \frac{1}{4}$$

With vs. without replacement

Sampling 20 instead of 5 students from the 20 student.

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

Replacement False:

$P(Avi) = 1$

Replacement True:

$P(Avi) < 1$

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

$$\binom{12}{4} = C(\underset{n}{12}, \underset{k}{4})$$

Order = False

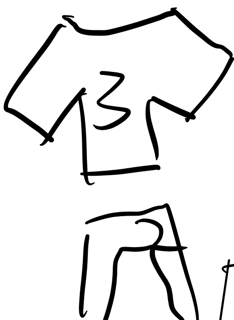
Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?

$\underbrace{C(5,2)}_{\text{of 5 markers choose 2}} * \underbrace{C(7,2)}_{\text{of 7 crayons choose 2}}$

$C(5,3) * C(7,1)$



3 t-shirt * 2 pants = 6 outfits

$C(5,3) = C(5,2)$

$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!\,k!}$ $C(n,n-k)$

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

S = all sets of 4 art supplies

$|S| = C(12,4)$

all equally likely.

Prob(at least 2 markers)

$= \dfrac{\text{\# ways to choose 4 art supplies such that at least 2 are Markers}}{C(12,4)}$

$= \dfrac{C(12,4) - [C(5,0)\cdot C(7,4) + C(5,1) * C(7,3)]}{C(12,4)}$

# Markers to select	
0	$C(5,0) * C(7,4)$
1	$C(5,1) * C(7,3)$
2	$C(5,2) * C(7,2)$
3	$C(5,3) * C(7,1)$
4	$C(5,4) * C(7,0)$

Fair coin

T H T

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

Question 3: Suppose we flip a **fair coin** 10 times.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$$\boxed{\left(\frac{1}{2}\right)^{10}} = \frac{1}{2^{10}}$$

$$\text{Prob}(\#H = \#T) = \frac{\#\text{ seq with } 5H, 5T}{\#\text{ seq with } 10\ H/T}$$

$$= \frac{C(10,5)}{2^{10}}$$

$k^n = 2^{10}$

$C(10,5)$ = # ways to choose 5 positions for the Hs

T T H T H T H H T H

$P(\text{single output}) = \frac{1}{2^{10}}$

Unfair coin

Question 4: Suppose we flip an **unfair coin** 10 times. The coin is biased such that for each flip, $P(\text{heads}) = \frac{1}{3}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

T H T T H T H H T H

$\frac{2}{3} \times \frac{1}{3} \times \frac{2}{3}$ - - - - - - - - -

How many T's? 5 $\rightarrow \left(\frac{2}{3}\right)^5$

How many H's? 5 $\rightarrow \left(\frac{1}{3}\right)^5$

$\left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^5$

$P(H) = 1/3$, flip 10 times, Prob(5H, 5T)

TTHTHTHTHTHTH

Prob $= (\frac{2}{3})^5(\frac{1}{3})^5$ TTTTTHHHHH

Prob $= (\frac{2}{3})^5(\frac{1}{3})^5$ · TTTTTTTTHHH

$(\frac{2}{3})^6(\frac{1}{3})^4$ · TTTTTTHHHH

· HHHHHHHHHH $(\frac{1}{3})^{10}$

S = all sequence of H's and T's

E: Event I care about

include all outcomes

w/ 5H & 5T

$(\frac{2}{3})^5(\frac{1}{3})^5$

Prob of event E

$$= \sum_{S \in E} \text{Prob}(S) = \sum_{S \in E} \left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^5$$

$$= C(10,5) \cdot \left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^5$$

Deck of cards

- There are 52 cards in a standard deck (4 suits, 13 values).

4 Suit 13 Values

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Sets of Card

→ order = False

Deck of cards

1. How many 5 card hands are there in poker?

$C(52,5)$

1st 2nd 3rd 4th 5th

$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$

2. How many 5 card hands are there where all cards are of the same suit (a **flush**)?

ex) ♡3, ♡Q, ♡A, ♡5, ♡10

1) What suit do we choose from? 4 options
2) which 5 cards from the suit?
$C(13,5)$

$4 \cdot C(13,5)$

3. How many 5 card hands are there that include a **four-of-a-kind** (four cards of the same value)?

ex) 5♢, 5♤, 5♡, 5♧, 8♢

1) which value to repeat? 13 options

2) what other cards? $12 \cdot 4 = 48$ options

$13 \cdot 12 \cdot 4$

4. How many 5 card hands are there that have a **straight** (all card values consecutive)?

ex) 5♡, 6♢, 7♤, 8♧, 9♡

1) which value can I pick as 1st card?
(2, 3, 4, 5, 6, 7, 8, 9, 10) → 9 options

$\underbrace{9}_{1st} \cdot \underbrace{4}_{2nd} \cdot \underbrace{4}_{3rd} \cdot \underbrace{4}_{4th} \cdot \underbrace{4}_{5th} = 9 \cdot 4^5$

5. How many 5 card hands are there that are a **straight flush** (all card values consecutive and of the same suit)?

6. How many 5 card hands are there that include exactly **one pair** (values aabcd)?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.
- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.
- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.