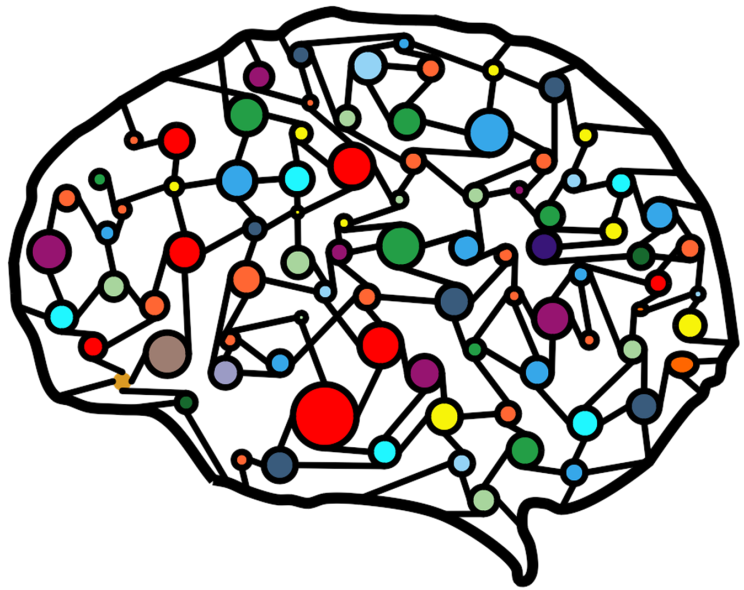
Lecture 19 - More Probability and Combinatorics Examples



DSC 40A, Winter 2024

Announcements

- Discussion is tonight.
- Homework 6 has been released, due **Wednesday at 11:59pm**.
- Homework 7 (last HW) will be released this Friday along with the second Extra Credit opportunity.

Agenda

- Lots of examples.

Selecting students — overview

We're going answer the same question using several different techniques.

All students are equally likely to be select

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

False

① Solve this with order → True.

P(n,k)

Give "names" to all students:

A B C D ..... T = 20 students

A → Avi

Selecting students (Method 1: using permutations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = a permutation (ordered selection) of 5 students chosen from A, B, ... T

ex) LPFGA, LFGAT, ...

$$P(\text{A included}) = \frac{\#\text{ permutation w/ A}}{\text{total \# of permutations}}$$

$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = P(20,5)$

Numerator: # of permutations including A.

ex). ACJTE

A _ _ _ _

1 × 19 × 18 × 17 × 16

→ 1 × 19 × 18 × 17 × 16

$\frac{5 \cdot P(19,4)}{P(20,5)} = \frac{5 \cdot \frac{19!}{15!}}{\frac{20!}{15!}} = 5 \cdot \frac{19!}{20!} = \frac{5}{20} = 1/4$

5 cases

A _ _ _ _

_ A _ _ _

_ _ A _ _

_ _ _ A _

_ _ _ _ A

total # of perm

is

$5 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

$= 5 \cdot P(19,4)$

Selecting students (Method 2: using permutations and the complement)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$\frac{\#\text{ Perms including A}}{\text{total \# Perms}} = \frac{\text{total \# of Perms} - \#\text{ Perms not include A}}{\text{total \# of Perms}}$$

$$\#\text{ perms not include A} = 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 = P(19,5)$$

$$\frac{P(20,5) - P(19,5)}{P(20,5)} = \frac{1}{4}$$

Selecting students (Method 3: using combinations)

Order = False

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = set of 5 students, chosen from A, B, ..., T

order = False.

ex) {B, D, G, H, M}

$$P(\text{A included}) = \frac{\#\text{ of sets of 5 students including A}}{\#\text{ of sets of 5 students}}$$

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

sets of 5 students: $C(20,5) = \binom{20}{5}$

$$= \frac{2!}{15!5!}$$

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Avi?

sets include Avi

$n = 19$

$k = 4$

$C(n,k) = C(19,4)$

$P(\text{A included}) = \dfrac{C(19,4)}{C(20,5)}$

$\frac{1}{4}$

$C(19,4)$: # of other students except A, B, C, ... T

$C(20,5)$: choose 4 other students to go w Avi

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 4: “the easy way”)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

Fair coin

Question 3: Suppose we flip a **fair coin** 10 times.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Unfair coin

Question 4: Suppose we flip an **unfair coin** 10 times. The coin is biased such that for each flip, $P(\text{heads}) = \frac{1}{3}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Deck of cards

- There are 52 cards in a standard deck (4 suits, 13 values).

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Deck of cards

1. How many 5 card hands are there in poker?

2. How many 5 card hands are there where all cards are of the same suit (a **flush**)?

3. How many 5 card hands are there that include a **four-of-a-kind** (four cards of the same value)?

4. How many 5 card hands are there that have a **straight** (all card values consecutive)?

5. How many 5 card hands are there that are a **straight flush** (all card values consecutive and of the same suit)?

6. How many 5 card hands are there that include exactly **one pair** (values aabcd)?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.
- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.
- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.