Lecture 21 – Independence

DSC 40A, Winter 2024

Announcements

- Homework 7 will be released today
  - The second EC challenge will also be released, but does not count toward homework grade.
  - Last homework of this course!
- Midterm 2 will be on March 13rd
  - More information on next week's lecture

Agenda

- Recap of Lecture 20.
- Independence.

Last time

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- A set of events $E_1, E_2, \ldots, E_k$ is a **partition** of S if each outcome in S is in exactly one E_i.
- The **Law of Total Probability** states that if A is an event and $E_1, E_2, \ldots, E_k$ is a partition of S, then

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \ldots + P(E_k) \cdot P(A|E_k)$$
$$= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i)$$

- **Bayes' Theorem** states that

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- We often re-write the denominator $P(A)$ in Bayes' Theorem using the Law of Total Probability.

Frequentist vs. Bayesian

- Frequentist do not assign probability to hypothesis
  - A day is 24 hour, but measurement comes with uncertainties, we can model the uncertainty
- Bayesian assigns probability to hypotheses
  - $P(B) = P(Hypothesis)$ **Prior**
  - $P(A|B) = P(Data|Hypothesis)$ **Likelihood**
  - $P(B|A) = P(Hypothesis|Data)$ **Posterior**
  - $P(A) = P(Data)$ **Evidence**

Discussion Question

Consider any two events A and B. Choose the expression that's equivalent to

$$P(B|A) + P(\bar{B}|A).$$

a) $P(A)$
b) $1 - P(B)$
c) $P(B)$
d) $P(\bar{B})$
e) 1

Example: prosecutor's fallacy[1]

A bank was robbed yesterday by one person. Consider the following facts about the crime:

- The person who robbed the bank wore Nikes.
- Of the 10,000 other people who came to the bank yesterday, only 10 of them wore Nikes.

The prosecutor finds the prime suspect, and states that "given this evidence, the chance that the prime suspect was not at the crime scene is 1 in 1,000".

1. What is wrong with this statement?

2. Find the probability that the prime suspect is guilty given only the evidence in the exercise.

[1] exercise from Theory Meets Data textbook by Ani Adhikari

Independence

Updating probabilities

- Bayes' theorem describes how to update the probability of one event, given that another event has occurred.

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

  - $P(B)$ can be thought of as the "prior" probability of B occurring, before knowing anything about A.

  - $P(B|A)$ is sometimes called the "posterior" probability of B occurring, given that A occurred.

- What if knowing that A occurred doesn't change the probability that B occurs? In other words, what if

$$P(B|A) = P(B)$$

Independent events

- A and B are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$P(B|A) = P(B) \qquad\qquad P(A|B) = P(A)$$

- Otherwise, A and B are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

Independent events

- **Equivalent definition:** A and B are independent events if

$$P(A \cap B) = P(A) \cdot P(B)$$

- To check if A and B are independent, use whichever is easiest:
  - $P(B|A) = P(B)$.
  - $P(A|B) = P(A)$.
  - $P(A \cap B) = P(A) \cdot P(B)$.

Mutual exclusivity and independence

Discussion Question

Suppose A and B are two events with non-zero probabilities. Is it possible for A and B to be both mutually exclusive and independent?

a) Yes

b) No

Example: Venn diagrams

For three events A, B, and C, we know that

- A and C are independent,
- B and C are independent,
- A and B are mutually exclusive,
- $P(A \cup C) = \frac{2}{3}$, $P(B \cup C) = \frac{3}{4}$, $P(A \cup B \cup C) = \frac{11}{12}$.

Find $P(A)$, $P(B)$, and $P(C)$.

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - A is the event that the first card is a heart.
  - B is the event that the second card is a club.

- If you draw the cards **with** replacement, are A and B independent?

- If you draw the cards **without** replacement, are A and B independent?

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw one card from a deck of 52.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).

- Are A and B independent?

Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

Example: breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

Summary

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- Two events A and B are **independent** when knowledge of one event does not change the probability of the other event.
- There are there equivalent definitions of independence:
  - $P(B|A) = P(B)$
  - $P(A|B) = P(A)$
  - $P(A \cap B) = P(A) \cdot P(B)$