# Lecture 22 – Independence and Conditional Independence

DSC 40A, Winter 2024

# Announcements

- Homework 7 released last Friday, due this upcoming Friday.
- I will release a mock midterm 2 today, more information about the $2^{nd}$ midterm on Wednesday lecture.
- We have normal discussion today, next Monday's discussion is converted to a review session.
- Great source of practice problems for recent content: stat88.org/textbook.
- Also check out the Probability Roadmap on the resources tab of the course website.

# Agenda

- Independence.
- Conditional independence.

# Independence

## Independent events

- $A$ and $B$ are **independent events** if one event occurring does not affect the chance of the other event occurring.

- To check if $A$ and $B$ are independent, use whichever is easiest:
  - $P(B|A) = P(B)$.

    or
  - $P(A|B) = P(A)$.

    or
  - $P(A \cap B) = P(A) \cdot P(B)$.

$\hookrightarrow P(A \cap B) = P(B|A)P(A)$

only true if A, B independent $\Rightarrow = P(B) \cdot P(A)$

# Example: cards

$P(B|A) = \frac{13}{51}$

$\neq$

$P(B) = \frac{13}{52}$

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - $A$ is the event that the first card is a heart.
  - $B$ is the event that the second card is a club.

- If you draw the cards **with** replacement, are $A$ and $B$ independent?

Yes $P(B) = \frac{13}{52} \Leftrightarrow P(B|A) = \frac{13}{52}$

- If you draw the cards **without** replacement, are $A$ and $B$ independent? Once you remove Heart.

No then there are only 51 card left

# Example: cards

B

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A   A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

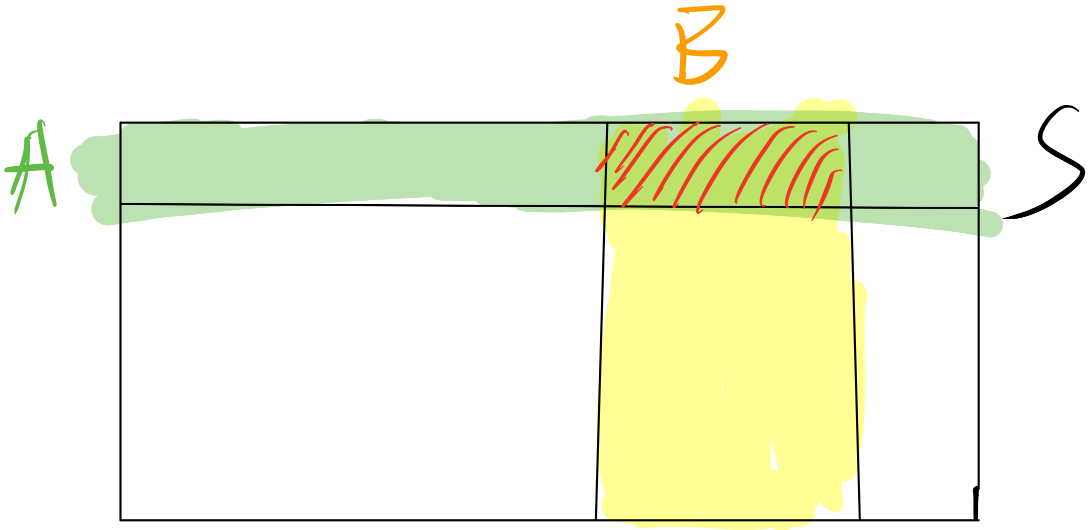
- Suppose you draw one card from a deck of 52.
  - $A$ is the event that the card is a heart.
  - $B$ is the event that the card is a face card (J, Q, K).
- Are $A$ and $B$ independent? Yes

$$P(B) = \frac{3}{13}$$

$$\|$$

$$P(B|A) = \frac{3}{13}$$

Venn Diagram: Visualizing independence when outcome are equally likely:

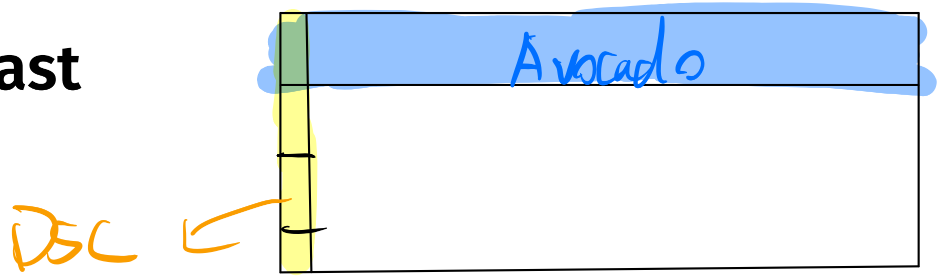



$P(B|A) = P(B)$ $\qquad$ $P(A|B) = P(A)$

# Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

# Example: breakfast




1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

$$P(Avo \mid DSC) = P(Avo) = 25\%$$

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

$$P(Avo \cap DSC) = P(Avo) \cdot P(DSC) = 1\% \cdot 25\%$$
$$= .25\%$$

# Conditional independence

# Conditional independence

- Sometimes, events that are dependent *become* independent, upon learning some new information.
- Or, events that are independent can become dependent, given additional information.

# Example: cards

B

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A  A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, ~~K~~, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
    - *A* is the event that the card is a heart.
    - *B* is the event that the card is a face card (J, Q, K).

- Are *A* and *B* independent? No

$$P(B|A) = \frac{3}{13} \neq P(B) = \frac{11}{51}$$

# Example: cards

Card is Red

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - $A$ is the event that the card is a heart.
  - $B$ is the event that the card is a face card (J, Q, K).

- Suppose you learn that the card is red. Are $A$ and $B$ independent given this new information? Yes

Given that the card is red:

$$P(B|A) = \frac{3}{13} = P(B) = \frac{6}{26} = \frac{3}{13}$$

# Conditional independence

- Recall that $A$ and $B$ are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

- $A$ and $B$ are **conditionally independent** given $C$ if

$$P((A \cap B)|C) = P(A|C) \cdot P(B|C)$$

Comes from defn. of regular independence but with "Given C" everywhere.

- Given that $C$ occurs, this says that $A$ and $B$ are independent of one another.

# Assuming conditional independence

- Sometimes we assume that events are conditionally independent to make calculations easier.
- Real-world events are almost never exactly conditionally independent, but they may be close.

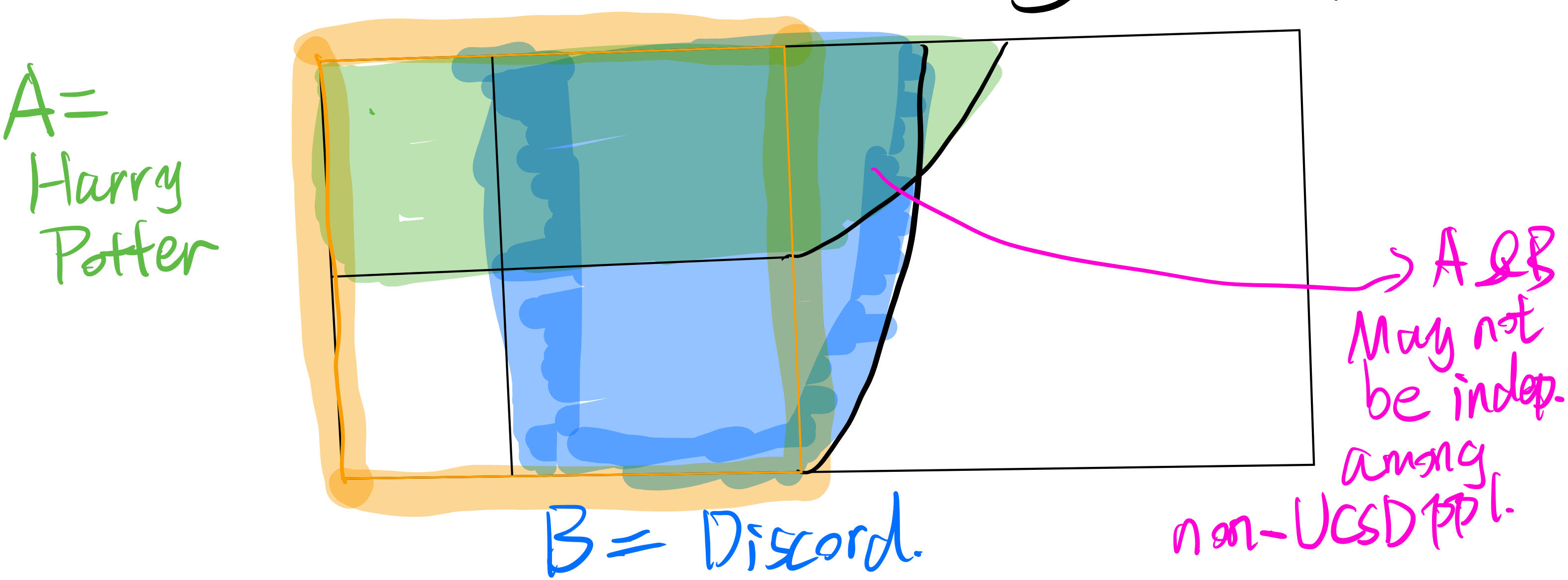
# Example: Harry Potter and Discord

$$P((A \cap B)|C) = P(A|C) * P(B|C) = 0.5 * 0.8 = 0.4$$

Suppose that 50% of UCSD students like Harry Potter and 80% of UCSD students use Discord. What is the probability that a random UCSD student likes Harry Potter and uses Discord, assuming that these events are conditionally independent given that a person is a UCSD student?

C = UCSD Student.

S = all people



# Independence vs. conditional independence

Yes
- Is it reasonable to assume conditional independence of
  - liking Harry Potter
  - using Discord

  given that a person is a UCSD student?

No
- Is it reasonable to assume independence of these events in general, among all people?

**Discussion Question**

Which assumptions do you think are reasonable?

a) Both
b) Conditional independence only
c) Independence (in general) only
d) Neither

# Independence vs. conditional independence

In general, there is **no relationship** between independence and conditional independence. All four scenarios below are possible.

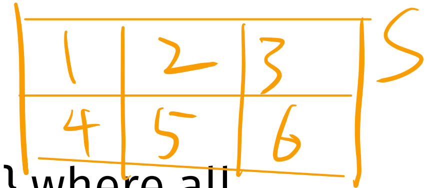
True True
- **Scenario 1:** $A$ and $B$ **are** independent. $A$ and $B$ **are** conditionally independent given $C$.

True False
- **Scenario 2:** $A$ and $B$ **are** independent. $A$ and $B$ **are not** conditionally independent given $C$.

False True
- **Scenario 3:** $A$ and $B$ **are not** independent. $A$ and $B$ **are** conditionally independent given $C$.

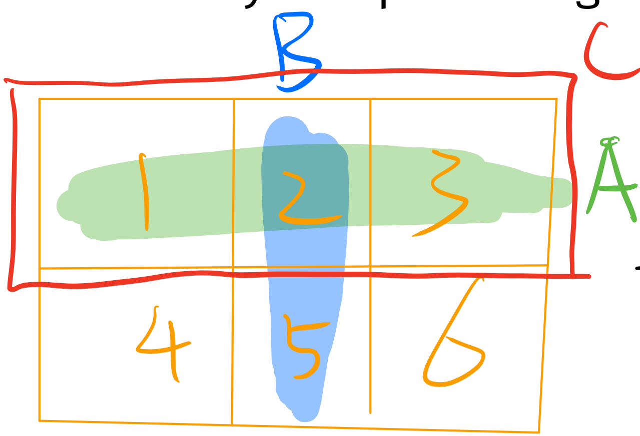
False False
- **Scenario 4:** $A$ and $B$ **are not** independent. $A$ and $B$ **are not** conditionally independent given $C$.

# Example: constructing events




- Consider a sample space $S = \{1, 2, 3, 4, 5, 6\}$ where all outcomes are equally likely.
- For each scenario, specify events $A$, $B$, and $C$ that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

**Scenario 1:** $A$ and $B$ **are** independent. $A$ and $B$ **are** conditionally independent given $C$.




$P(A \mid B) = \frac{1}{2} = P(A) = \frac{1}{2}$

with condition C:

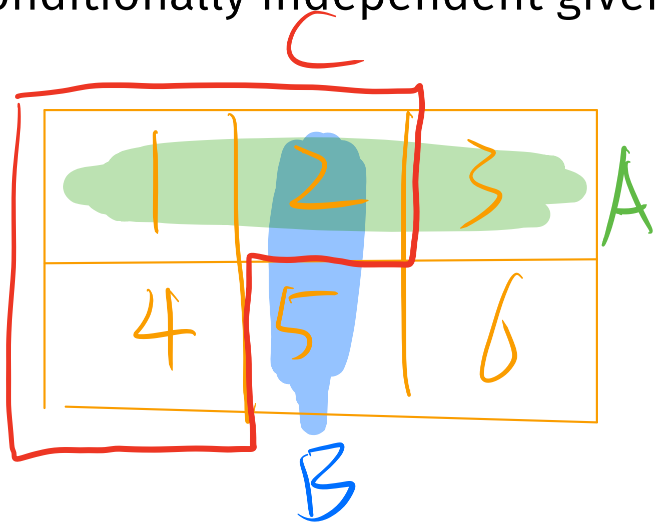
$P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$

$\frac{1}{3} = 1 \cdot \frac{1}{3}$

# Example: constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6\}$ where all outcomes are equally likely.
- For each scenario, specify events $A$, $B$, and $C$ that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

**Scenario 2:** $A$ and $B$ **are** independent. $A$ and $B$ **are not** conditionally independent given $C$.




Already know A,B ind.

Check if A B are conditionally ind.

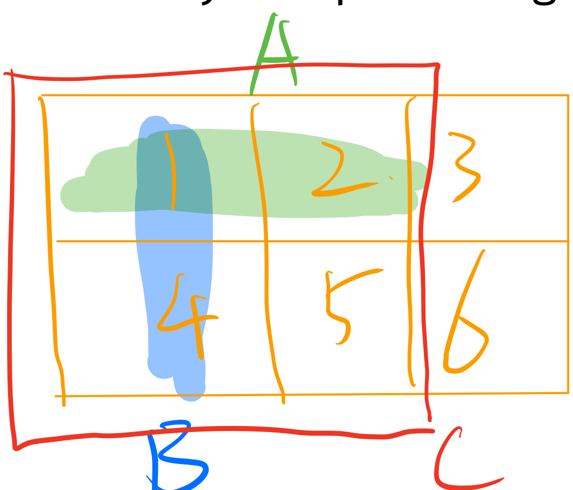
$$P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

$$\frac{1}{3} \neq \frac{2}{3} \cdot \frac{1}{3}$$

# Example: constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6\}$ where all outcomes are equally likely.
- For each scenario, specify events $A$, $B$, and $C$ that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

**Scenario 3:** $A$ and $B$ **are not** independent. $A$ and $B$ **are** conditionally independent given $C$.




Independence:

$$P(A \cap B) = P(A) \cdot P(B)$$

$$\frac{1}{6} \neq \frac{1}{3} \cdot \frac{1}{3}$$

Cond. Indep.:

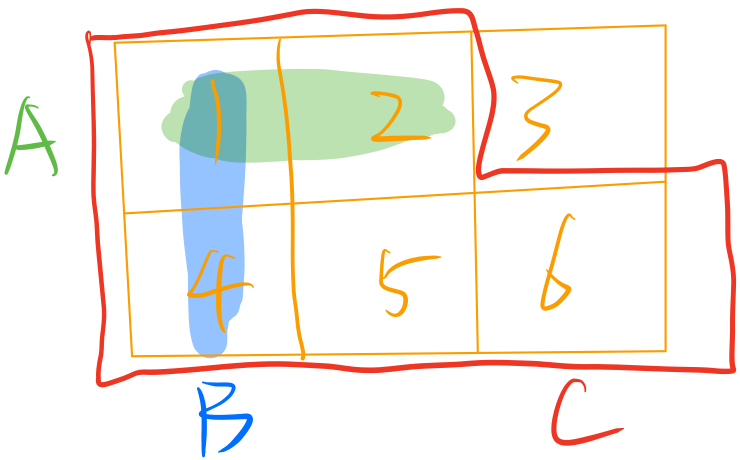
$$P(A \cap B | C) = P(A|C) \cdot P(B|C)$$

$$\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$$

# Example: constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6\}$ where all outcomes are equally likely.
- For each scenario, specify events $A$, $B$, and $C$ that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

**Scenario 4:** $A$ and $B$ **are not** independent. $A$ and $B$ **are not** conditionally independent given $C$.




A & B are not indep.

Show A&B are not conditionally indep.

$$P((A \cap B)|C) = P(A|C) \cdot P(B|C)$$

$$\frac{1}{5} \neq \frac{2}{5} \cdot \frac{2}{5}$$

# Summary

## Summary

- Two events $A$ and $B$ are **independent** when knowledge of one event does not change the probability of the other event.
    - Equivalent conditions: $P(B|A) = P(B)$, $P(A|B) = P(A)$, $P(A \cap B) = P(A) \cdot P(B)$.
- Two events $A$ and $B$ are **conditionally independent** if they are independent given knowledge of a third event, $C$.
    - Condition: $P((A \cap B)|C) = P(A|C) \cdot P(B|C)$.
- In general, there is no relationship between independence and conditional independence.
- **Next time:** Using Bayes' theorem and conditional independence to solve the **classification problem** in machine learning.