Lecture 23 – Naive Bayes

DSC 40A, Winter 2024

Announcements

- **Midterm 2 is Wednesday 3/13** during lecture.
- I'm travelling from next Tuesday to Saturday, Prof. Gal Mishne will proctor the midterm on 3/13.
- Next week we have two review sessions, one is Monday discussion (for Midterm 2), one is Friday lecture (for Final)
  - Zhenduo (TA) and tutors will lead both review sessions.
- Final is on March 22, Final part I/Part II is replacable with midterm 1/midterm 2, respectively.

About Midterm 2

- You'll be allowed an unlimited number of handwritten note sheets for Midterm 2. Start studying and preparing your notes now!
  - Has to be handwritten, no printed notes.

- Midterm 2 covers lecture 13-24. Clustering is included, but the vast majority will be probability and combinatorics.

- No calculators.
  - There will be some numerical calculations, but no very hard ones.

- Assigned seats will be posted on Campuswire.

- We will not answer questions during the exam. State your assumptions if anything is unclear.

Midterm 2 Preparation Strategy

- One useful strategy is attributing complicated real-world problems into known models.
  - Example: rolling a die
- Unlike Part I of this course which is mostly proof, in Part II we have done lots of examples in lecture, make sure you understand them. If not, please ask questions in OH/Campuswire.
  - You will see something similar in the exam.
- Everything I covered in the lecture 13-24 is possible to appear in the midterm.

Midterm vs. Final

- The **majority** of Midterm I and II are long-answer homework-style questions, which would require explanation and be graded with partial credit.
  - **Pro:** partial credits; **Con:** you will have to show your work

- Final Part I and II will be **mostly** multiple choice or filling in the numerical answer.
  - **Pro:** you don’t need to justify your answer; **Con:** No partial credits

- Extra Credit applies to midterms only, not final.

Agenda

- Classification.
- Classification and conditional independence.
- Naive Bayes.

Recap: Bayes' theorem, independence, and conditional independence

- Bayes' theorem: $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$.
- A and B are **independent** if $P(A \cap B) = P(A) \cdot P(B)$.

One of the 3 equation of indp.

- A and B are **conditionally independent** given C if $P((A \cap B)|C) = P(A|C) \cdot P(B|C)$.
  - In general, there is no relationship between independence and conditional independence.

$$P(A,B|C) \rightarrow \cap$$

$$= P((A \cap B)|C)$$

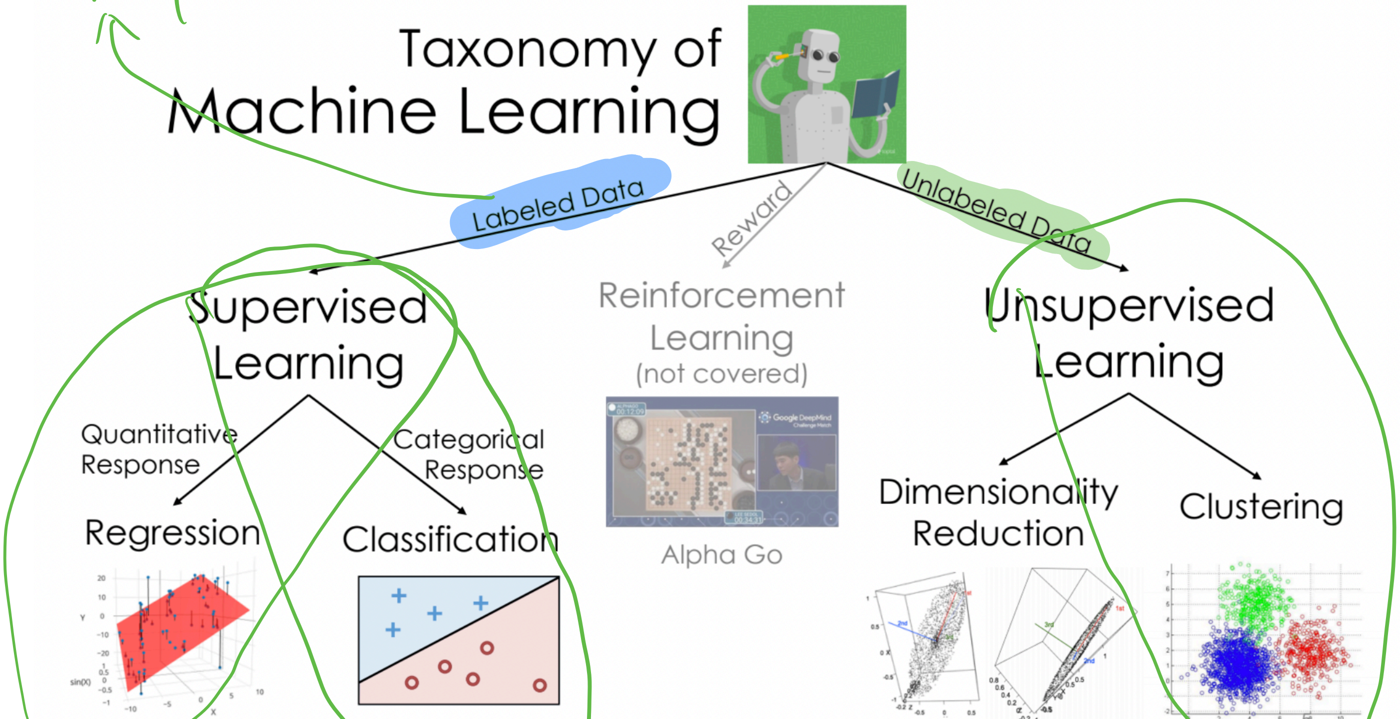
Classification

$$P(A,B|C) = P(A|B,C) \cdot P(B|C)$$

$$\frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{\cancel{P(B \cap C)}} \cdot \frac{\cancel{P(B \cap C)}}{P(C)}$$

know value of response variable

Taxonomy of Machine Learning[1]



Predicting DS salary
y= salary

y= ripe/unripe

[1] taken from Joseph Gonzalez at UC Berkeley

Classification problems

- Like with regression, we're interested in making predictions based on data (called **training data**) for which we know the value of the response variable.

  y

- The difference is that the response variable is now **categorical**.
- Categories are called **classes**.
- Example classification problems:
  - Deciding whether a patient has kidney disease.
  - Identifying handwritten digits.
  - Determining whether an avocado is ripe.
  - Predicting whether credit card activity is fraudulent.

Example: avocados

feature

You have a green-black avocado, and want to know if it is ripe.

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

Training Data

Question: Based on this data, would you predict that your avocado is ripe or unripe?

$P(\text{ripe} \mid \text{green-black})$

$\frac{3}{5}$ ← 3 riped green-black
← 5 green-black total

$P(\text{unripe} \mid \text{green-black})$

$\frac{2}{5}$ ← 2 unriped
← 5 green-black

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

Strategy: Calculate two probabilities:

$$P(\text{ripe}|\text{green-black})$$

$$P(\text{unripe}|\text{green-black})$$

Then, predict the class with a **larger** probability.

$$\frac{3}{5} > \frac{2}{5}$$

Estimating probabilities

- We would like to determine $P(\text{ripe}|\text{green-black})$ and $P(\text{unripe}|\text{green-black})$ for all avocados in the universe.

- All we have is a single dataset, which is a **sample** of all avocados in the universe.

- We can estimate these probabilities by using sample proportions.

$$P(\text{ripe}|\text{green-black}) \approx \frac{\overbrace{\text{\# ripe green-black avocados in sample}}^{=3}}{\underbrace{\text{\# green-black avocados in sample}}_{=5}}$$

- Per the **law of large numbers** in DSC 10, larger samples lead to more reliable estimates of population parameters.

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$P(\text{ripe}|\text{green-black}) = \frac{3}{5}$

$P(\text{unripe}|\text{green-black}) = \frac{2}{5}$

direct interpretation

Bayes' theorem for classification

- Suppose that A is the event that an avocado has certain features, and B is the event that an avocado belongs to a certain class. Then, by Bayes' theorem:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- More generally:

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

- What's the point?
  - Usually, it's not possible to estimate $P(\text{class}|\text{features})$ directly from the data we have.
  - Instead, we have to estimate $P(\text{class})$, $P(\text{features}|\text{class})$, and $P(\text{features})$ separately.

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

Using Bayes Theorem

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$$P(\underbrace{\text{class}}_{\text{ripe}}|\underbrace{\text{features}}_{\text{green-black}}) = \frac{P(\text{class})\cdot P(\text{features}|\text{class})}{P(\text{features})}$$

$$= \frac{P(\text{ripe}) \cdot P(\text{green-black}|\text{ripe})}{P(\text{green-black})}$$

$$= \frac{\frac{\cancel{7}}{\cancel{11}} \cdot \frac{3}{\cancel{7}}}{\frac{5}{\cancel{11}}} = \frac{3}{5}$$

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

$$P(\text{unriped}|\text{green-black})$$

$$= \frac{P(\text{unriped}) \cdot P(\text{green-black}|\text{unripe})}{P(\text{green-black})}$$

$$= \frac{\frac{\cancel{4}}{\cancel{11}} \cdot \frac{2}{\cancel{4}}}{\frac{5}{\cancel{11}}} = \frac{2}{5}$$

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

means 'proportional to'

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

Shortcut: Both probabilities have the same denominator. The larger one is the one with the larger numerator.

$P(\text{ripe}|\text{green-black})$

$\propto P(\text{ripe}) \cdot P(\text{green-black}|\text{ripe})$

$P(\text{unripe}|\text{green-black})$

$\propto P(\text{unripe}) \cdot P(\text{green-black}|\text{unripe})$

Classification and conditional independence

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

Strategy: Calculate $P(\text{ripe}|\text{features})$ and $P(\text{unripe}|\text{features})$ and choose the class with the **larger** probability.

$$P(\text{ripe}|\text{firm, green-black, Zutano})$$

$$P(\text{unripe}|\text{firm, green-black, Zutano})$$

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

Issue: We have not seen a firm green-black Zutano avocado before.

This means that $P(\text{ripe}|\text{firm, green-black, Zutano})$ and $P(\text{unripe}|\text{firm, green-black, Zutano})$ are undefined.

A simplifying assumption

- We want to find $P(\text{ripe}|\text{firm, green-black, Zutano})$, but there are no firm green-black Zutano avocados in our dataset.

- Bayes' theorem tells us this probability is equal to

$P((A \cap B \cap C)|E)$

$$P(\text{ripe}|\text{firm, green-black, Zutano}) = \frac{P(\text{ripe}) \cdot P(\text{firm, green-black, Zutano}|\text{ripe})}{P(\text{firm, green-black, Zutano})}$$

- **Key idea: Assume** that features are **conditionally independent** given a class (e.g. ripe).

$$P(\text{firm, green-black, Zutano}|\text{ripe}) = P(\text{firm}|\text{ripe}) \cdot P(\text{green-black}|\text{ripe}) \cdot P(\text{Zutano}|\text{ripe})$$

ignore calculating denom.

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

$$P(\text{ripe}|\text{firm, green-black, Zutano}) = \frac{P(\text{ripe}) \cdot P(\text{firm, green-black, Zutano}|\text{ripe})}{P(\text{firm, green-black, Zutano})}$$

$$\propto P(\text{ripe}) \cdot P(\text{firm}|\text{ripe}) \cdot P(\text{green-black}|\text{ripe}) \cdot P(\text{Zutano}|\text{ripe})$$

$$= \frac{7}{11} \cdot \frac{1}{7} \cdot \frac{3}{7} \cdot \frac{2}{7} \approx \frac{6}{539}$$

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

$$P(\text{unripe}|\text{firm, green-black, Zutano}) = \frac{P(\text{unripe}) \cdot P(\text{firm, green-black, Zutano}|\text{unripe})}{P(\text{firm, green-black, Zutano})}$$

$$\propto P(\text{unripe}) \cdot P(\text{firm}|\text{unripe}) \cdot P(\text{green-bl}|\text{unripe}) \cdot P(\text{Zut}|\text{unrip}$$

$$\frac{4}{11} \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{2}{4} = \frac{3}{44} = \frac{6}{88}$$

Conclusion

- The numerator of $P(\text{ripe}|\text{firm, green-black, Zutano})$ is $\frac{6}{539}$.
- The numerator of $P(\text{unripe}|\text{firm, green-black, Zutano})$ is $\frac{6}{88}$.
  - Both probabilities have the same denominator, $P(\text{firm, green-black, Zutano})$.
  - Since we're just interested in seeing which one is larger, we can ignore the denominator and compare numerators.
- Since the numerator for unripe is **larger** than the numerator for ripe, we **predict that our avocado is unripe**.

Naive Bayes

Naive Bayes classifier

- We want to predict a class, given certain features.
- Using Bayes' theorem, we write

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

ripe / unripe

- For each class, we compute the numerator using the **naive assumption of conditional independence of features given the class**.
- We estimate each term in the numerator based on the training data.
- We predict the class with the largest numerator.
  - Works if we have multiple classes, too!

Dictionary

Definitions from Oxford Languages · Learn more

na·ive

adjective

(of a person or action) showing a lack of experience, wisdom, or judgment.
"the rather naive young man had been totally misled"

- (of a person) natural and unaffected; innocent.
  "Andy had a sweet, naive look when he smiled"

  Similar: innocent · unsophisticated · artless · ingenuous · inexperienced

- of or denoting art produced in a straightforward style that deliberately rejects sophisticated artistic techniques and has a bold directness resembling a child's work, typically in bright colors with little or no perspective.

Example: avocados, again

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

Naive Bayes Assumption

You have a soft green-black Hass avocado. Based on this data, would you predict that your avocado is ripe or unripe?

$$P(\text{ripe} \mid \text{soft}, \text{g-b}, \text{Hass}) \propto P(\text{ripe})\, P(\text{soft} \mid \text{ripe}) \cdot P(\text{g-b} \mid \text{ripe}) \cdot P(\text{Hass} \mid \text{ripe})$$

$$\frac{7}{11} \cdot \frac{4}{7} \cdot \frac{3}{7} \cdot \frac{5}{7}$$

$$P(\text{unripe} \mid \text{soft}, \text{g-b}, \text{Hass}) \propto P(\text{unripe}) \cdot P(\text{soft} \mid \text{unripe}) \cdot P(\text{g-b} \mid \text{unripe}) \cdot P(\text{Hass} \mid \text{unripe})$$

$$= \frac{4}{11} \cdot \frac{0}{4} \cdots = 0$$

Uh oh...

- There are no soft unripe avocados in the data set.

- The estimate $P(\text{soft}|\text{unripe}) \approx \frac{\#\text{ soft unripe avocados}}{\#\text{ unripe avocados}}$ is 0.

- The estimated numerator,
$P(\text{unripe}) \cdot P(\text{soft, green-black, Hass}|\text{unripe}) = P(\text{unripe}) \cdot P(\text{soft}|\text{unripe}) \cdot P(\text{green-black}|\text{unripe}) \cdot P(\text{Hass}|\text{unripe})$, is also 0.

- But just because there isn't a soft unripe avocado in the data set, doesn't mean that it's impossible for one to exist!

- **Idea:** Adjust the numerators and denominators of our estimate so that they're never 0.

Smoothing

- **Without** smoothing:

$$P(\text{soft}|\text{unripe}) \approx \frac{\#\text{ soft unripe}}{\#\text{ soft unripe} + \#\text{ medium unripe} + \#\text{ firm unripe}}$$

$$P(\text{medium}|\text{unripe}) \approx \frac{\#\text{ medium unripe}}{\#\text{ soft unripe} + \#\text{ medium unripe} + \#\text{ firm unripe}}$$

$$P(\text{firm}|\text{unripe}) \approx \frac{\#\text{ firm unripe}}{\#\text{ soft unripe} + \#\text{ medium unripe} + \#\text{ firm unripe}}$$

- **With** smoothing:

$$P(\text{soft}|\text{unripe}) \approx \frac{\#\text{ soft unripe} + 1}{\#\text{ soft unripe} + 1 + \#\text{ medium unripe} + 1 + \#\text{ firm unripe} + 1}$$

$$P(\text{medium}|\text{unripe}) \approx \frac{\#\text{ medium unripe} + 1}{\#\text{ soft unripe} + 1 + \#\text{ medium unripe} + 1 + \#\text{ firm unripe} + 1}$$

$$P(\text{firm}|\text{unripe}) \approx \frac{\#\text{ firm unripe} + 1}{\#\text{ soft unripe} + 1 + \#\text{ medium unripe} + 1 + \#\text{ firm unripe} + 1}$$

- When smoothing, we add 1 to the count of every group whenever we're estimating a conditional probability.

Example: avocados, with smoothing

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a soft green-black Hass avocado. Using Naive Bayes, **with smoothing**, would you predict that your avocado is ripe or unripe?

Summary

Summary

- In classification, our goal is to predict a discrete category, called a **class**, given some features.

- The Naive Bayes classifier works by estimating the numerator of $P(\text{class}|\text{features})$ for all possible classes.

- It uses Bayes' theorem:

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

- It also uses a simplifying assumption, that features are conditionally independent given a class:

$$P(\text{features}|\text{class}) = P(\text{feature}_1|\text{class}) \cdot P(\text{feature}_2|\text{class}) \cdot \ldots$$