DSC 40A Fall 2025 - Group Work Session 1

due Monday, September 29th at 11:59PM

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **tag all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive full credit, you must work in a group of two to four students for at least 50 minutes in your assigned discussion section. You can also self-organize a group and meet outside of discussion section for 80 percent credit. You may not do the groupwork alone.

1 Objects in Linear Algebra

Problem 1.

Let $n, d \geq 1$ be fixed positive integers. For each subproblem, answer with one of the following choices:

- a scalar
- a vector in $\mathbb{R}^d$
- a vector in $\mathbb{R}^n$
- a $d \times d$ matrix
- a $d \times n$ matrix
- an $n \times n$ matrix
- an $n \times d$ matrix

a) For each $i = 1, ..., d$, let $\vec{x}^{(i)}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\sum_{i=1}^{d} \vec{x}^{(i)^T} \vec{x}^{(i)}$$

b) For each $i = 1, ..., d$, let $\vec{x}^{(i)}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\sum_{i=1}^{d} \vec{x}^{(i)} \vec{x}^{(i)^T}$$

c) Let $\vec{x}$ be a vector in $\mathbb{R}^n$, and let A be an $n \times n$ matrix. What type of object is:

$$\vec{x}^T A \vec{x}$$

d) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\frac{\vec{x}}{\|\vec{x}\|}$$

e) Let $\vec{x}$ be a vector in $\mathbb{R}^n$, and let A be a $d \times n$ matrix. What type of object is:

$$\frac{A\vec{x}}{\|\vec{x}\|} + (\vec{x}^T A^T A\vec{x})A\vec{x}$$

f) Let A be a $d \times n$ matrix. Suppose $A^T A$ is invertible. What type of object is:

$$(A^T A)^{-1}$$

Problem 2.

Let $p \geq 1$ be a fixed positive integer and let $x, y \in \mathbb{R}^p$ be fixed **nonzero** vectors. Writing x in column vector notation we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \tag{1}$$

and similarly for y. Of the following ten expressions, five are provably equal (meaning, they are all equal to each other **regardless** of x, y). Circle them and eliminate the remaining five "odd ones out" which are not provably equal (meaning, they are not equal to the other expressions in general unless we "know" x, y).

1. xy^T.
2. $\|y\|x^T\widehat{y}$ where $\|y\|$ is the length of y and $\widehat{y}$ is the unit vector parallel to y.
3. $\sum_{i=1}^{p-1} x_i y_{i+1}$.
4. $\left(\sum_{i=1}^{p} (x_i y_i)^3\right)^{1/3}$
5. $\sum_{i=3}^{p+2} x_{i-2} y_{i-2}$.
6. $(O^T x)^T (O^T y)$, where $O \in \mathbb{R}^{p \times p}$ is an orthogonal matrix satisfying $O^T O = \mathrm{Id}_{p \times p}$.
7. $\frac{1}{2}(Ax)^T y$, where $A \in \mathbb{R}^{p \times p}$ is any matrix for which y^T is a left eigenvector with eigenvalue 2.
8. $(U^{-1}x)^T U y$ where $U \in \mathbb{R}^{p \times p}$ is any nonsingular (i.e. invertible) matrix.
9. $\frac{1}{\|x\|^2} \mathrm{Tr}\left(y^T (yx^T) x\right)$.
10. $\frac{1}{\|x\|^2} \mathrm{Tr}\left(yx^T xx^T\right)$.

2 Matrix Multiplication

This is intended to help review some concepts from MATH 18 related to matrix-vector and matrix-matric multiplication.

Problem 3.

Let's brush up on our matrix-vector multiplication skills. Suppose we have a matrix and a vector defined as follows:

$$X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 1 & -2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

Evaluate $X\vec{w}$.

Problem 4.

Perhaps you noticed something while computing $X\vec{w}$ in the above problem. In particular, you may recall from MATH 18 that the matrix-vector multiplication, $X\vec{w}$, is a linear combination of the columns of the matrix, X, by the appropriate weights from the vector, $\vec{w}$.

Fill in each blank below with a single number using the numbers from Problem 4.

$$X\vec{w} = \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} + \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} + \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

Problem 5.

Now, let's generalize this concept. Let X be an $n \times d$ matrix, such that each column, $\vec{x}^{(i)}$ is a vector in $\mathbb{R}^n$. Let $\vec{w}$ be a vector in $\mathbb{R}^d$. Fill in the blanks:

$$X\vec{w} = \sum_{i=1}^{\square} \square$$

3 Subspaces and Bases

This section contains some practice problems to help review the concepts of vector subspaces and vector space bases.

Problem 6.

Start here: What are the three criteria for a subset S of some vector space V to qualify as a vector **subspace**?

Problem 7.

For each of the following scenarios, determine whether the provided subset S qualifies as a vector subspace. *Bonus: What is the dimension of S in the case(s) where it is a vector subspace?*

a) $V = \mathbb{R}^n$ and S is the set of all $x \in V$ such that $x^T \mathbf{1}_n = 0$ where $\mathbf{1}_n$ is the vector of all ones.

b) $V = \mathbb{R}^n$ and S is the set of all $x \in V$ such that $\sum_{i=1}^{n} x_i = -2$.

c) $V = \mathbb{R}^{n \times n}$ and S is the set of all matrices $A \in V$ such that $A^2 = \mathrm{A}$

d) $V = \mathbb{R}^{n \times n}$ and S is the set of all matrices $A \in V$ such that $BA = 0_{n \times n}$ for some matrix B.

Problem 8.

Another quick check: What are the three criteria for a subset B of some vector space V to qualify as a **basis** for V?

Problem 9.

Let e_i be the i-th standard basis vector which is one at index i and zero otherwise. Which of the following sets $\{x_i\}_{i=1}^{n}$ form a basis of $\mathbb{R}^n$?

a) $x_i = e_i - e_1$ for $1 \leq i \leq n$.

b) $x_i = e_i + \mathbf{1}_n$ for $1 \leq i \leq n$ where $\mathbf{1}_n$ is the vector of all ones. *Hint: Check this for case $n = 2$ and $n = 3$ to get started.*

c) $x_i = ie_i$. In other words, $x_1 = e_1, x_2 = 2e_2, x_3 = 3e_3$, and so on.