DSC 40A - Homework 6

due Friday, November 21st at 11:59 PM

Homeworks are due to Gradescope by 11:59PM on the due date.

You can use a slip day to extend the deadline by 24 hours; you have four slip days to use in total throughout the quarter.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain and justify** your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Homeworks should be written up and turned in by each student individually. You may talk to other students in the class about the problems and discuss solution strategies, but you should not share any written communication and you should not check answers with classmates. You can tell someone how to do a homework problem, but you cannot show them how to do it. **Only handwritten solutions will be accepted (use of tablets is permitted). Do not typeset your homework (using LaTeXor any other software)**.

For each problem you submit, you should **cite your sources** by including a list of names of other students with whom you discussed the problem. Instructors do not need to be cited.

This homework will be graded out of **69 points**. The point value of each problem or sub-problem is indicated by the number of avocados shown.

Problem 1. Mid-quarter Feedback Form

Make sure to fill out this Mid-quarter Reflection and Feedback Form, linked here! Note that this form is anonymous so that you feel comfortable sharing constructive feedback. Everyone will be given two points and the course staff is trusting you to fill out the form.

Problem 2. Avi's Bootstraps

Recall from DSC 10 the process of bootstrap resampling. From a population of size n, we draw one random sample of size k, without replacement. Then, we create many bootstrap resamples by sampling k elements from the original sample, with replacement.

Suppose we have a population of 100 stuffed toys, one of which is Avi. From this population, we draw a sample of size 20, without replacement. From this original sample, we create 5 different bootstrap resamples.

a) What is the probability that Avi is included in the original sample?

b) What is the probability that Avi is included in the first resample?

Hint: Don't make any assumptions about whether Avi was included in the original sample.

c) What is the probability that Avi is included in some resample?

Problem 3. Amped Up

Congratulations — you've won a month's supply of energy drinks! Each week, you receive **one prize box with 8 drinks** containing an assortment of drinks from companies like Monster, Red Bull, Celsius, etc. Each week's box is a surprise; you don't know which drinks it will contain until you open it.

For this problem, assume:

- There are 50 different possible energy drinks (equivalently, 50 different companies).
- Each of the 50 drinks is manufactured in equal quantities, so you are no more likely to get any one drink than any other.
- Each company produces only one type of energy drink, so (for instance) all drinks produced by Monster are the same, all drinks produced by Red Bull are the same, and so on.

Thus, saying that you have drinks from "different companies" is the same as saying that you have different *types* of drinks.

Note: Your final answers can include $P(n,k)$, $C(n,k)$, or $\binom{n}{k}$ without simplifying further.

a) Suppose that the contents of each week's prize box are selected uniformly and independently at random **without replacement** from among the 50 possible drinks (so each box contains 8 distinct drinks). If you take 2 weeks' worth of prize boxes (i.e., 16 total energy drinks), what is the probability that you end up with drinks from exactly 8 different companies?

b) Suppose that the contents of each week's prize box are selected uniformly and independently at random **without replacement** from among the 50 possible drinks. If you take 2 weeks' worth of prize boxes (i.e., 16 total energy drinks), what is the probability that you end up with drinks from exactly 16 different companies?

c) Now suppose that the contents of each week's prize box are selected uniformly and independently at random **with replacement** from among the 50 possible drinks (so repeats are allowed, even within a single prize box). If you take 2 weeks' worth of prize boxes (i.e., 16 total energy *drinks*), what is the probability that you end up with drinks from exactly 16 different companies?

d) Suppose that the contents of each week's prize box are selected uniformly and independently at random **with replacement** from among the 50 possible drinks. If you take 2 weeks' worth of prize boxes (i.e., 16 total energy drinks), what is the probability that you end up with drinks from exactly 15 different companies?

Problem 4. Pascal's Identity

I have invited 5 friends over for dinner. Suppose I have 11 plates in the cupboard, each of which is a different color, and I want to create a combination of 6 of them. From lecture, we know that this can be done in $\binom{11}{6}$ ways. In this problem, we'll look at another way to arrive at this same result.

a) The purple plate is my favorite. How many ways can I select 6 plates from my collection of 11 plates such that my purple plate is one of the 6 selected?

b) My friend Varun hates purple. How many ways can I select 6 plates from my collection of 11 plates such that my purple plate is **not** one of the 6 selected?

c) Using the results of the previous two parts, how many ways can I select 6 plates from my collection of 11 plates?

Note: You must write your answer in terms of your results to parts (a) and (b); you will get no credit if you write $\binom{11}{6}$.

d) What you've just discovered is an application of Pascal's Identity, which states that

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Prove Pascal's identity.

Hint: Start with the left-hand side, use the definition of the binomial coefficient, and try and bring both terms to a common denominator. Another hint — what is $\frac{n}{n!}$?

You may assume $n > k$. *Note however, that in general, there is no restriction on the relative sizes of n and k. If $n < k$, the value of the binomial coefficient is zero and the identity remains valid.*

Side note: Pascal's identity has a close connection to Pascal's triangle.

e) Using Pascal's rule, show Hockey-stick identity:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \ldots + \binom{k}{k}$$

The Hockey-stick name comes from the visualized pattern in Pascal's triangle, in which each element's value equals the sum of its left-upper and right-upper neighbors:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$\binom{4}{2} = \binom{3}{1} + \binom{2}{1} + \binom{1}{1}$$

Problem 5. seven ate nine

Note: Your finals solutions can include $P(n, k), C(n, k)$ or $\binom{n}{k}$ without simplifying further.

a) How many ways are there to arrange the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ so that the first four numbers in the arrangement are in increasing order and the last five numbers in the arrangement are in decreasing order? For example, 136987542 is one such arrangement.

b) How many ways are there to arrange the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ so that the first four numbers in the arrangement are even? For example, 428617593 is one such arrangement.

c) How many ways are there to arrange the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ so that the arrangement includes the numbers 789 together and in order? For example, 789531246 and 123456789 are such arrangements.

d) For a given collection of numbers, define a "setsequence" of length n to be a sequence of two elements, where the first element is a set (a set is unordered) of length n and the second is a sequence of length n. If we use curly braces $\{\}$ to denote a set and square brackets [ ] to denote a sequence, then an example of a setsequence of length 3 for the collection $1, 2, 3, 4$ is $[\{1, 2, 3\}, [4, 1, 3]]$. How many different setsequences of length 2 are there for the collection $1, 2, 3, 4$?

Problem 6. House of Cards

You have a standard deck of cards containing 52 cards. There are 13 cards in each of 4 suits (hearts $\heartsuit$, spades $\spadesuit$, diamonds $\diamondsuit$, and clubs $\clubsuit$). Within a suit, the 13 cards each have a different rank. In ascending order, these ranks are

$$2, 3, 4, 5, 6, 7, 8, 9, 10, \text{Jack}, \text{Queen}, \text{King}, \text{Ace}.$$

You are playing a four-player card game using **two** regular decks of cards. The two decks are first shuffled together, making a combined deck of 104 cards, and then dealt out so that each player receives 26 cards. Two identical cards (same rank and same suit, one from each deck) are called a *pair*. A pair can only be outranked (beaten) by a pair of higher-ranked cards *within the same suit*.

a) Suppose both decks of cards are first shuffled together and then dealt out to the four players. What is the probability that *you* get a pair of Kings of Hearts (i.e., both Kings of Hearts)?

First, express your answer using $P(n, k)$ notation and interpret the solution in terms of ordered choices (permutations). Next, rewrite the expression using $C(n, k)$ or $\binom{n}{k}$ notation and interpret the solution in terms of unordered choices (combinations).

b) Suppose both decks of cards are first shuffled together and then dealt out to the four players. You are dealt two Kings of Hearts. What is the probability that *some other player* (one of the other three players) has a pair of Aces of Hearts?

c) Suppose both decks of cards are first shuffled together and then dealt out to the four players. You are dealt two Kings of Hearts and you do **not** have any Aces of Hearts. What is the probability that some other player has a pair of Aces of Hearts?

d) Suppose both decks of cards are first shuffled together and then dealt out to the four players. What is the probability that you get *either* a pair of Kings of Hearts *or* a pair of Aces of Hearts (possibly both)?

Problem 7. Bundling up!

Ava has a drawer containing exactly ten mittens: one pair each of plain, dotted, striped, plaid, and zigzag mittens.

a) On Monday morning, Ava selects two mittens at random from the drawer. What is the probability that the mittens are a matching pair?

b) After wearing the mittens on Monday she places them in a basket separate from the drawer. On Tuesday morning she selects two mittens at random from the eight remaining in the drawer. What is the probability that the mittens match on Tuesday? *Note: Your answer should not depend on whether the mittens matched on Monday.*

c) On Tuesday evening Ava does laundry, so on Wednesday morning all ten mittens are back in the drawer (independent of previous days). What is the probability that she wears at least one *zigzag* mitten on Monday, Tuesday, **and** Wednesday? *Show all your calculations.*