Lecture 3

Empirical Risk Minimization - mean absolute error

DSC 40A, Fall 2025

Agenda

- Recap: Mean squared error.
- Another loss function.
- Minimizing mean absolute error.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

The modeling recipe

We've implicitly introduced a three-step process for finding optimal model parameters (like h^*) that we can use for making predictions:

1. Choose a model.

constant model $H(x) = h$ Another choice : $H(x) = w_0 + w_1 x$

2. Choose a loss function.

$L_{sq}(y_i, h) = (y_i - h)^2$ changing today

3. Minimize average loss to find optimal model parameters.

$h^* = \text{Mean}\{y_1, y_2, \ldots, y_n\}$ different h^*?

Recap: Mean squared error

Minimizing using calculus

We'd like to minimize:

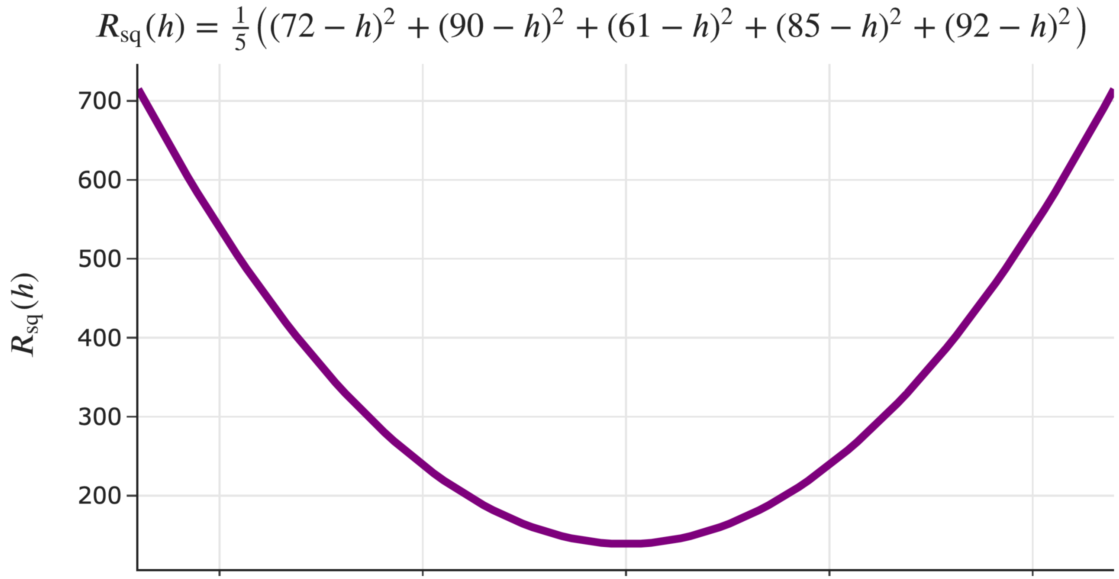
$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} \overbrace{(y_i - h)^2}^{\ell(y_i, h)}$$

$$\frac{d}{dh} \ell_{sq}(y_i, h) = 2(h - y_i)$$

In order to minimize $R_{\text{sq}}(h)$, we:

1. take its derivative with respect to h, $\longrightarrow \frac{d}{dh} R_{sq}(h) = \frac{1}{n} \sum_{i=1}^{n} \frac{d}{dh} \ell_{sq}(y_i, h)$
2. set it equal to 0, $\longrightarrow \frac{2}{n} \sum_{i=1}^{n} (h - y_i) = 0$
3. solve for the resulting h^*, and $\longrightarrow h^* = \text{mean}\{y_1, \ldots, y_n\}$
4. perform a second derivative test to ensure we found a minimum.

Step 4: Second derivative test



We already saw that $R_{\mathrm{sq}}(h)$ is **convex**, i.e. that it opens upwards, so the h^* we found must be a minimum, not a maximum.

$$\overbrace{1+1+\ldots+1}^{n \text{ times}}$$

$$\frac{d^2}{dh^2} R_{sq}(h) = \frac{d}{dh}\left[\frac{2}{n}\sum_{i=1}^{n}(h-y_i)\right] = \frac{2}{n}\sum_{i=1}^{n}\frac{d}{dh}h = \frac{2}{n}\sum_{i=1}^{n}1 = \frac{2}{n}\cdot n = 2 > 0$$

$$\frac{d^2}{dh^2} R_{sq}(h) > 0 \implies h^* \text{ is a minimizer}$$

The mean minimizes mean squared error!

- The problem we set out to solve was, find the h^* that minimizes:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

- The answer is:

$$h^* = \text{Mean}(y_1, y_2, \dots, y_n)$$

- The **best constant prediction**, in terms of mean squared error, is always the **mean**.
- This answer is always unique!
- We call h^* our **optimal model parameter**, for when we use:
  - the constant model, $H(x) = h$, and
  - the squared loss function, $L_{\text{sq}}(y_i, h) = (y_i - h)^2$.

Bonus: the mean is easy to compute

```
def mean(numbers):
    total = 0
    for number in numbers:
        total = total + number
    return total / len(numbers)
```

- Time complexity $\Theta(n)$

40B

Aside: Notation

Another way of writing

$$h^* \text{ is the value of } h \text{ that minimizes } \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

is

"the arguement that minimizes"

$$h^* = \underset{h}{\operatorname{argmin}} \left(\frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2 \right)$$

h^* is the solution to an **optimization problem**.

Another loss function

Another loss function

- Last lecture, we started by computing the **error** for each of our predictions, but ran into the issue that some errors were positive and some were negative.

$$e_i = y_i - H(x_i)$$

data we have

prediction

- The solution was to **square** the errors, so that all are non-negative. The resulting loss function is called **squared loss**.

$$L_{\text{sq}}(y_i, H(x_i)) = (y_i - H(x_i))^2$$

- Another loss function, which also measures how far $H(x_i)$ is from y_i, is **absolute loss**.

$$L_{\text{abs}}(y_i, H(x_i)) = |y_i - H(x_i)|$$

Squared loss vs. absolute loss

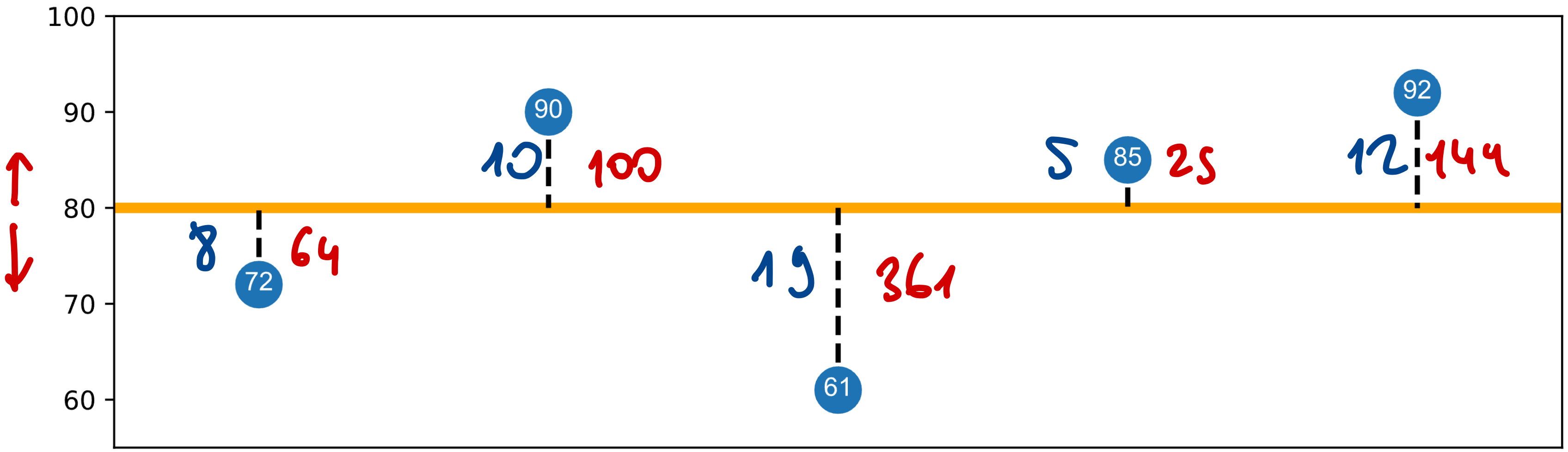
For the constant model, $H(x_i) = h$, so we can simplify our loss functions as follows:

- Squared loss: $L_{\text{sq}}(y_i, h) = (y_i - h)^2$.
- Absolute loss: $L_{\text{abs}}(y_i, h) = |y_i - h|$.

mean

Consider, again, our example dataset of five commute times and the prediction $h = 80$.

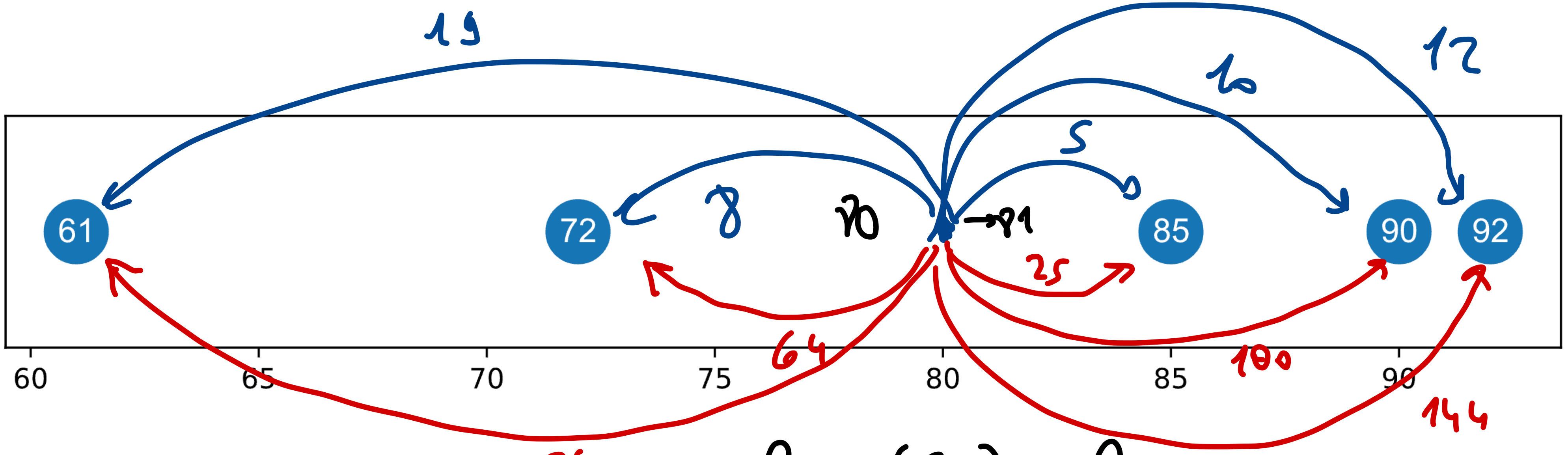
$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 92$



Squared loss vs. absolute loss

80 minimizes avg. sqr loss
but not necessarily avg abs. loss

- When we use squared loss, h^* is the point at which the average **squared** loss is minimized.
- When we use absolute loss, h^* is the point at which the average **absolute** loss is minimized.



$R_{abs}(81) < R_{abs}(80)$

Mean absolute error

- Suppose we collect n commute times, $y_1, y_2, \ldots, y_n$.
- The **<u>average</u> absolute loss**, or **<u>mean</u> absolute error (MAE)**, of the prediction h is:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} \overbrace{|y_i - h|}^{l_{\text{abs}}(y_i, h)}$$

- We'd like to find the best prediction, h^*.
- Previously, when using squared loss we used calculus to find the optimal model parameter h^* that minimized R_{sq}.
- Can we use calculus to minimize $R_{\text{abs}}(h)$, too?

Minimizing mean absolute error

Minimizing using calculus, again

We'd like to minimize:

known

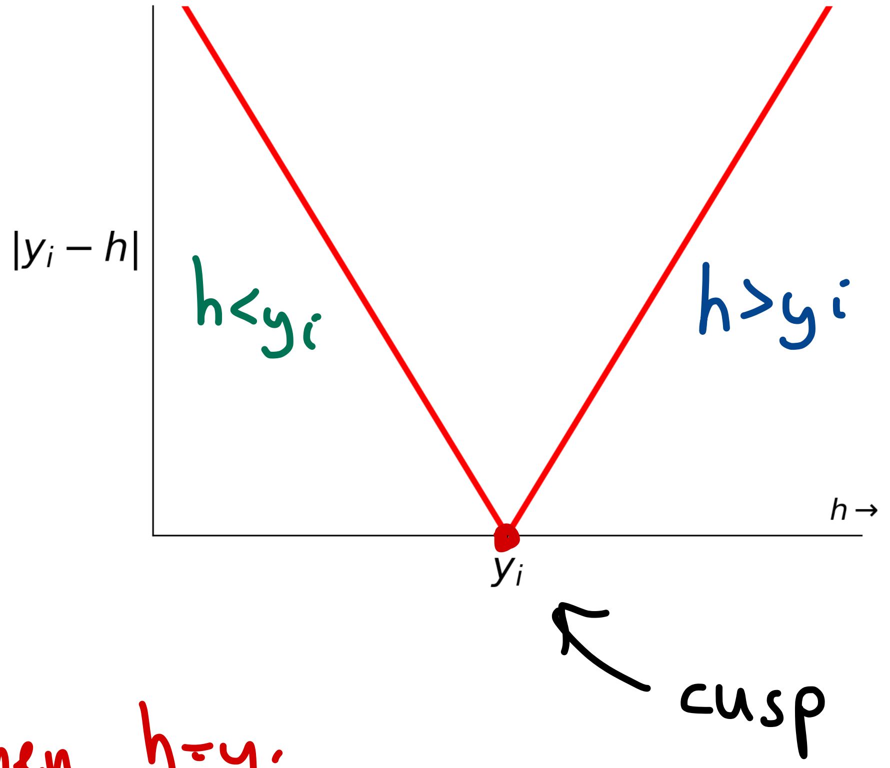
unknown

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

In order to minimize $R_{\text{abs}}(h)$, we:

1. take its derivative with respect to h,
2. set it equal to 0,
3. solve for the resulting h^*, and
4. perform a second derivative test to ensure we found a minimum.

Step 0: The derivative of $|y_i - h|$



when $h=y_i$:
$|h-y_i|=|y_i-h|=0$

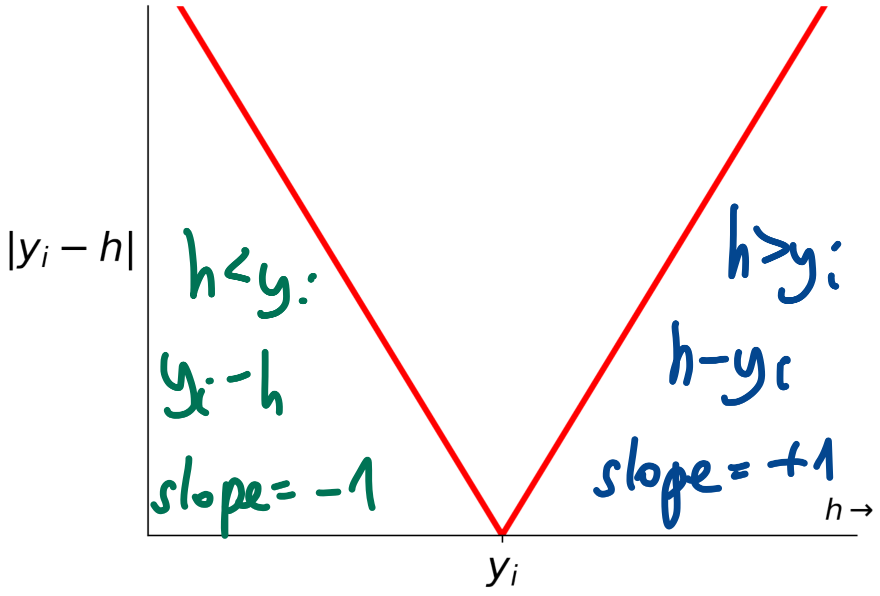
Remember that $|x|$ is a **piecewise linear** function of x:

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

So, $|y_i - h|$ is also a piecewise linear function of h:

$$|y_i - h| = \begin{cases} y_i - h & h < y_i \\ 0 & y_i = h \\ h - y_i & h > y_i \end{cases}$$

Step 0: The "derivative" of $|y_i - h|$



$$|y_i - h| = \begin{cases} y_i - h & h < y_i \\ 0 & y_i = h \\ h - y_i & h > y_i \end{cases}$$

What is $\frac{d}{dh}|y_i - h|$?

$$\frac{d}{dh}|y_i - h| = \begin{cases} -1 & h < y_i \\ \text{undefined} & h = y_i \\ +1 & h > y_i \end{cases}$$

ignore for now

Step 1: The "derivative" of $R_{\text{abs}}(h)$

$$\frac{d}{dh} |y_i - h| = \begin{cases} -1, & y_i > h \\ \text{undefined}, & y_i = h \\ +1, & y_i < h \end{cases}$$

$$\frac{d}{dh} R_{\text{abs}}(h) = \frac{d}{dh}\left(\frac{1}{n}\sum_{i=1}^{n} |y_i - h|\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{d}{dh} |y_i - h|$$

→ this a sum of -1, $+1$

we add $+1$ when $h > y_i$

we add -1 when $h < y_i$

$$= \frac{1}{n}\left[\underbrace{\#(h > y_i)}_{\text{num of datapoints } h \text{ is right of}} - \underbrace{\#(h < y_i)}_{\text{num of points } h \text{ is to the left of}}\right]$$

Ex: 61 72 85 90 91

$h = 80$

$$\frac{d}{dh} R_{\text{abs}}(80) = \frac{+1 + 1 - 1 - 1 - 1}{5} = -\frac{1}{5}$$

Question 🤔

Answer at q.dsc40a.com

The slope of R_{abs} at h is

$$\frac{1}{n}[(\# \text{ of } y_i < h) - (\# \text{ of } y_i > h)]$$

Suppose that the number of points n is odd. At what value of h does the slope change from negative to positive?

- A) h = mean of $\{y_1, \ldots, y_n\}$
- B) h = median of $\{y_1, \ldots, y_n\}$
- C) h = mode of $\{y_1, \ldots, y_n\}$

Steps 2 and 3: Set to 0 and solve for the minimizer, h^*

$$\frac{d}{dh} R_{abs}(h) = \frac{1}{n}\left[\#(h > y_i) - \#(h < y_i)\right] = 0$$

$$\#(h > y_i) = \#(h < y_i)$$

We want h^* is the value when the

number of datapoints to the left of h

=

number of datapoints to the right of h

median!

The median minimizes mean absolute error!

- The new problem we set out to solve was, find the h^* that minimizes:

MAE

↳ absolute

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

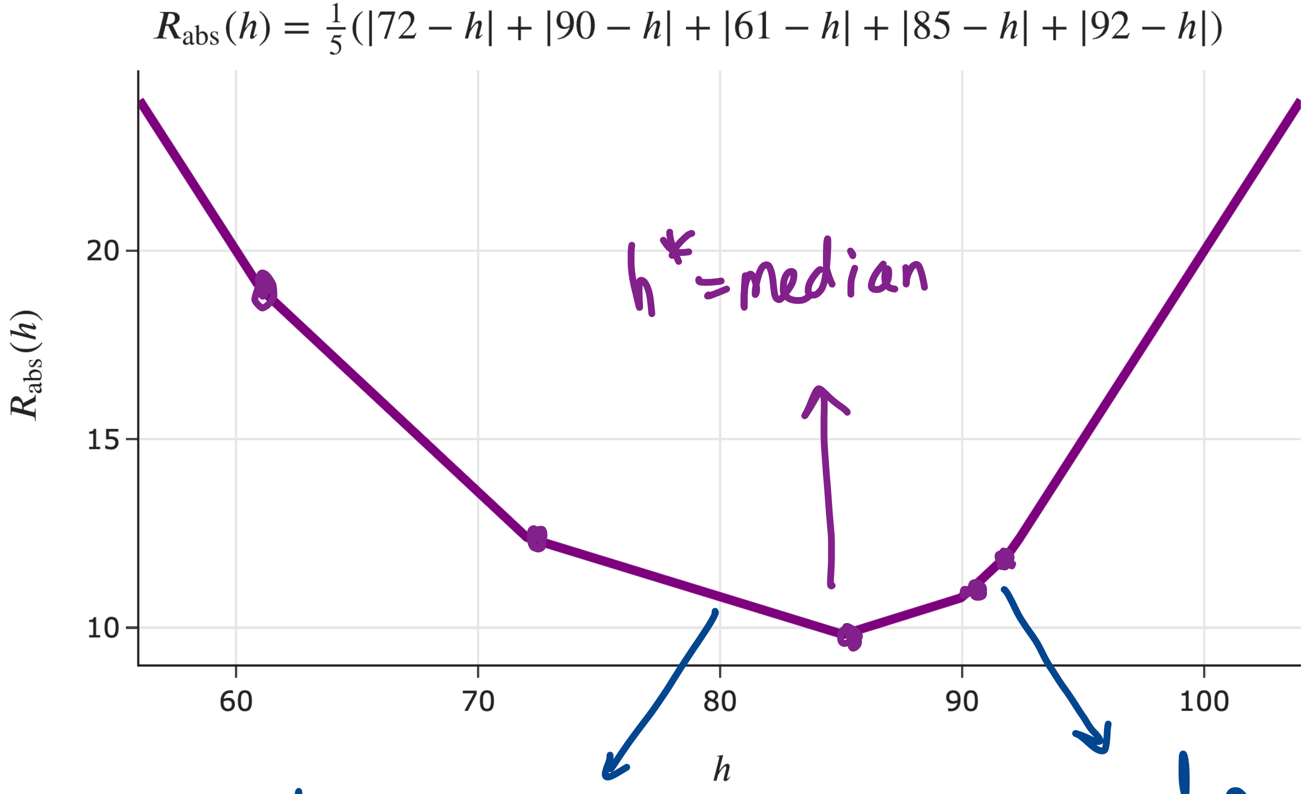
- The answer is:

$$h^* = \text{Median}(y_1, y_2, \ldots, y_n)$$

- This is because the median has an equal number of data points to the left of it and to the right of it.
- To make a bit more sense of this result, let's graph $R_{\text{abs}}(h)$.

Visualizing mean absolute error

$= \frac{1}{5}\left[\underset{72}{V} + \underset{61}{V} + \underset{85}{V} + \underset{90}{V} + \underset{92}{V} \right]$



$\frac{d}{dh} R_{abs}(80) = \frac{2-3}{5} = -\frac{1}{5}$

$\frac{d}{dh} R_{abs}(91) = \frac{4-1}{5} = \frac{3}{5}$

Consider, again, our example dataset of five commute times.

$$72, 90, 61, 85, 92$$

Where are the "bends" in the graph of $R_{\text{abs}}(h)$ – that is, where does its slope change?

Question 🤔

Answer at q.dsc40a.com

Consider, again, our example dataset of five commute times.

$$72, 90, 61, 85, 92$$

Suppose we add a sixth point so that our data is now

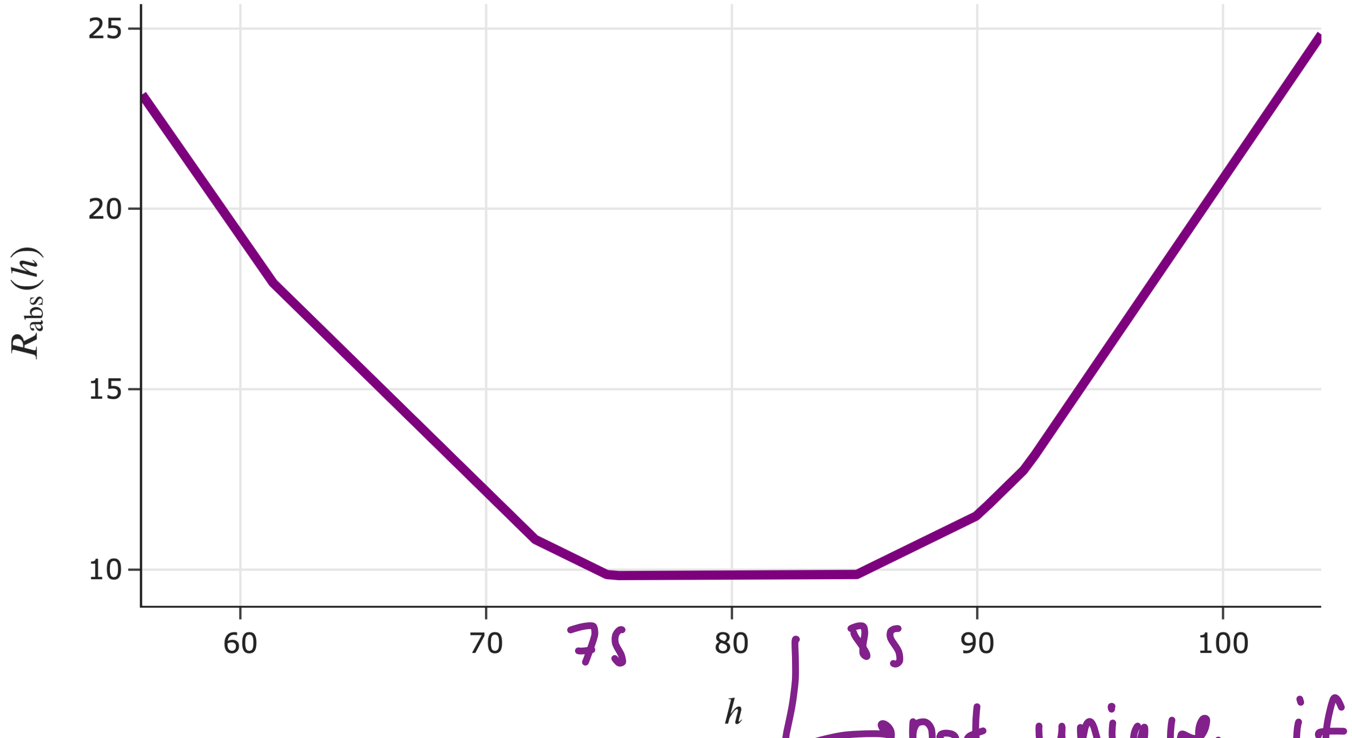
$$72, 90, 61, 85, 92, 75$$

Which of the following correctly describes the $h*$ that minimizes mean absolute error for our new dataset?

- A) 85 only
- B) 75 only
- C) 80 only
- D) Any value between 75 and 85 inclusive

Visualizing mean absolute error, with an even number of points

$R_{\text{abs}}(h) = \frac{1}{6}(|72 - h| + |90 - h| + |61 - h| + |85 - h| + |92 - h| + |75 - h|)$



What if we add a sixth data point?

$$72, 90, 61, 85, 92, 75$$

Is there a unique h^*?

$R_{abs}(h^*) = R_{abs}(h)$ for $75 \le h \le 85$

not unique if n is even

any $75 \le h \le 85$ minimizes $R_{abs}(h)$

The median minimizes mean absolute error!

- The new problem we set out to solve was, find the h^* that minimizes:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

- The answer is:

$$h^* = \text{Median}(y_1, y_2, \ldots, y_n)$$

The **best constant prediction**, in terms of mean absolute error, is always the **median**.

- When n is odd, this answer is unique.
- When n is even, any number between the middle two data points (when sorted) also minimizes mean absolute error.
- When n is even, define the median to be the mean of the middle two data points.

The modeling recipe, again

We've now made two full passes through our "modeling recipe."

1. Choose a model.

2. Choose a loss function.

3. Minimize average loss to find optimal model parameters.

Empirical risk minimization

- The formal name for the process of minimizing average loss is **empirical risk minimization**.
- Another name for "average loss" is **empirical risk**.
- When we use the squared loss function, $L_{\text{sq}}(y_i, h) = (y_i - h)^2$, the corresponding empirical risk is mean squared error:

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$$

- When we use the absolute loss function, $L_{\text{abs}}(y_i, h) = |y_i - h|$, the corresponding empirical risk is mean absolute error:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

Empirical risk minimization, in general

Key idea: If $L(y_i, h)$ is **any** loss function, the corresponding empirical risk is:

$$R(h) = \frac{1}{n} \sum_{i=1}^{n} L(y_i, h)$$

Question 🤔

Answer at q.dsc40a.com

What questions do you have?

Summary, next time

- $h^* = \text{Mean}(y_1, y_2, \ldots, y_n)$ minimizes mean squared error, $R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$.
- $h^* = \text{Median}(y_1, y_2, \ldots, y_n)$ minimizes mean absolute error, $R_{\text{abs}}(h) = \frac{1}{n}\sum_{i=1}^{n}|y_i - h|$.
- $R_{\text{sq}}(h)$ and $R_{\text{abs}}(h)$ are examples of **empirical risk** – that is, average loss.
- **Next time**: What's the relationship between the mean and median? What is the significance of $R_{\text{sq}}(h^*)$ and $R_{\text{abs}}(h^*)$?