Lectures 8-10

Linear algebra: Dot products and Projections

DSC 40A, Fall 2025

Question 🤔

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Dot product: coordinate definition

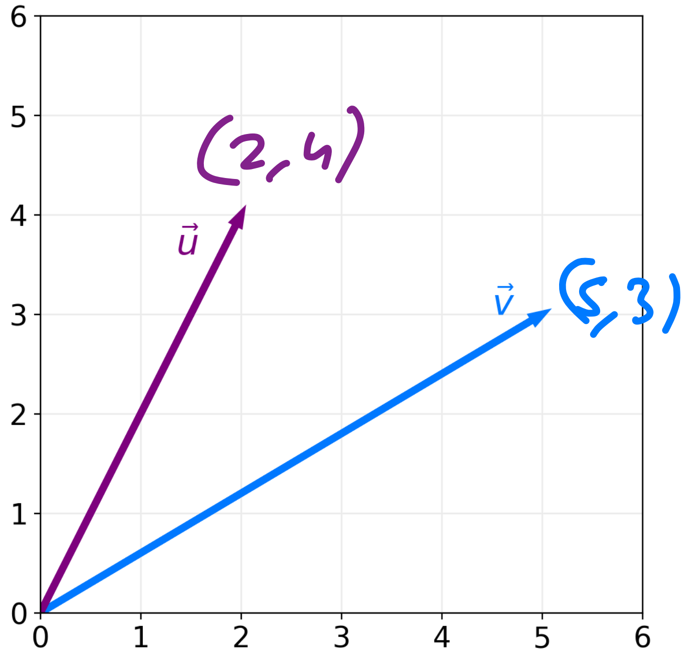
- The **dot product** of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is written as:

$$\vec{u}\cdot\vec{v} = \vec{u}^\top\vec{v} = \langle \vec{u}, \vec{v} \rangle$$

- The computational definition of the dot product:

$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The result is a **scalar**, i.e. a single number.



$$\vec{u}\cdot\vec{v} = 5\cdot 2 + 3\cdot 4 = 10 + 12 = 22 \in \mathbb{R}$$

$$\vec{u}^T\vec{v} = [5 \quad 3]\begin{bmatrix}2\\4\end{bmatrix} = 10 + 12 = 22 \in \mathbb{R}$$

$$f(\vec{u},\vec{v}) \in \mathbb{R}$$

$\mathbb{R}^n$ $\mathbb{R}^n$

Dot product: geometric definition

- The computational definition of the dot product:

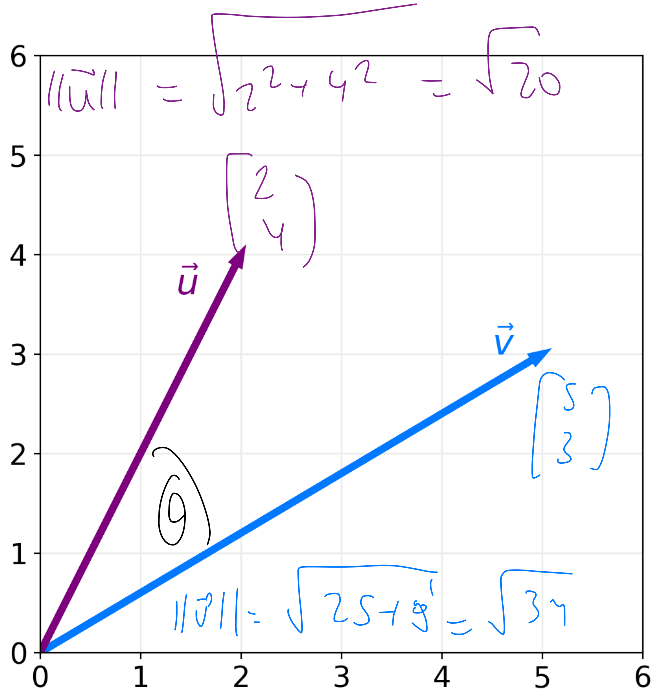
$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The geometric definition of the dot product:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

- The two definitions are equivalent! This equivalence allows us to find the angle θ between two vectors.

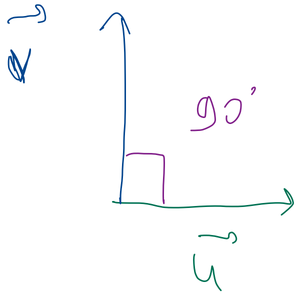


$\vec{u} \cdot \vec{v} = 22$ (from previous slides)

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \frac{22}{\sqrt{34}\cdot\sqrt{20}} = \frac{11}{\sqrt{34}\cdot\sqrt{5}} = \frac{11}{\sqrt{170}}$$

Orthogonal vectors

$\vec{u} \cdot \vec{v} = 0$



- Recall: $\cos 90^{\text{o}} = 0$.
- Since $\vec{u} \cdot \vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta$, if the angle between two vectors is 90^{o}, their dot product is $\|\vec{u}\|\,\|\vec{v}\| \cos 90^{\text{o}} = 0$.
- If the angle between two vectors is 90^{o}, we say they are perpendicular, or more generally, **orthogonal**.
- **Key idea**: $\vec{u} \perp \vec{v}$

$$\boxed{\text{two vectors are } \textbf{orthogonal} \iff \vec{u} \cdot \vec{v} = 0}$$

$\|\vec{u}\|, \|\vec{v}\| \neq 0$

if and only if

Exercise

Find a non-zero vector in $\mathbb{R}^3$ orthogonal to:

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -8 \end{bmatrix} \qquad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\vec{u} \cdot \vec{v} = 2u_1 + 5u_2 - 8u_3 = 0$$

Infinite possibilities!

$$w = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} \qquad \vec{v} \cdot \vec{w} = 2 \cdot 5 - 2 \cdot 5 = 0$$

$$z = \begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix} \qquad \vec{z} \cdot \vec{v} = 0 \cdot 2 + 8 \cdot 5 + 5 \cdot (-8) = 0$$

Spans and projections

Adding and scaling vectors

- The sum of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the **element-wise sum** of their components:

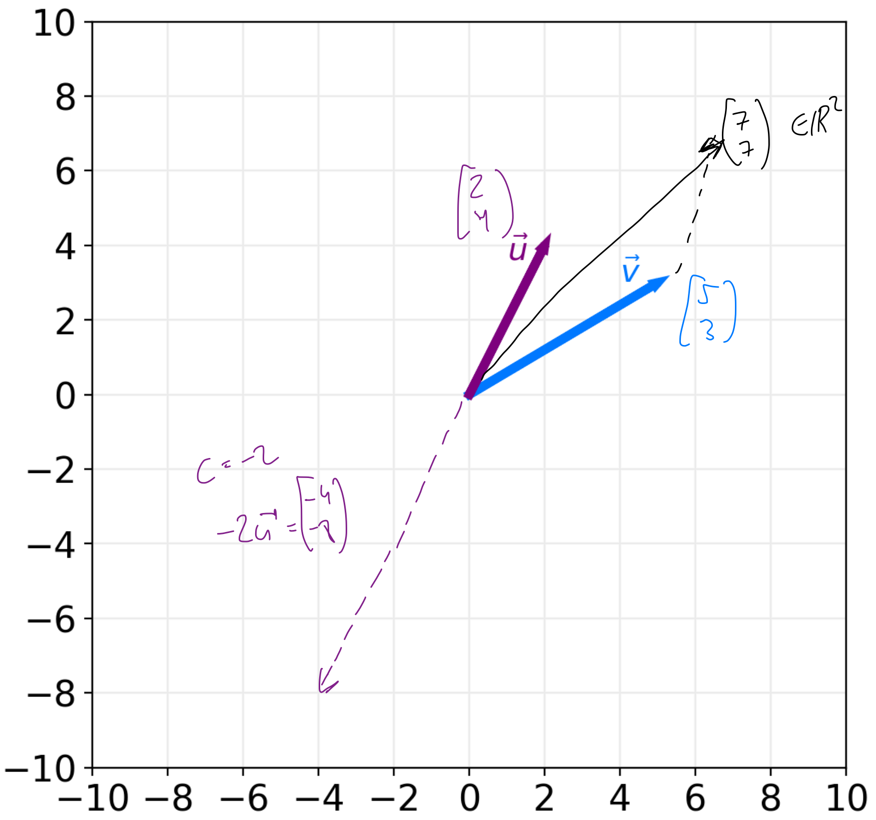
$\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \in \mathbb{R}^n$$

- If c is a scalar, then:

$c \in \mathbb{R}$

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$



Linear combinations

Let $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_d$ all be vectors in $\mathbb{R}^n$.

A **linear combination** of $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_d$ is any vector of the form:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d \in \mathbb{R}^n$$

$$= \sum_{i=1}^{d} a_i \vec{v}_i$$

where a_1, a_2, ..., a_d are all scalars.

$a_i \in \mathbb{R}$ for all $1 \le i \le d$

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^2$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$

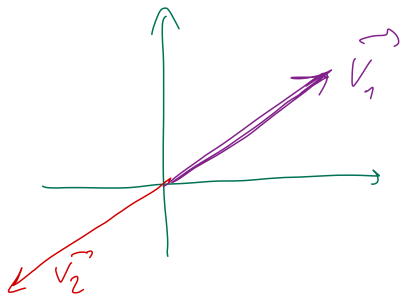
$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 \in \mathbb{R}^2$

$= \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Span

- Let $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_d$ all be vectors in $\mathbb{R}^n$.
- The **span** of $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_d$ is the set of all vectors that can be created using linear combinations of those vectors.
- Formal definition:

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d : a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$



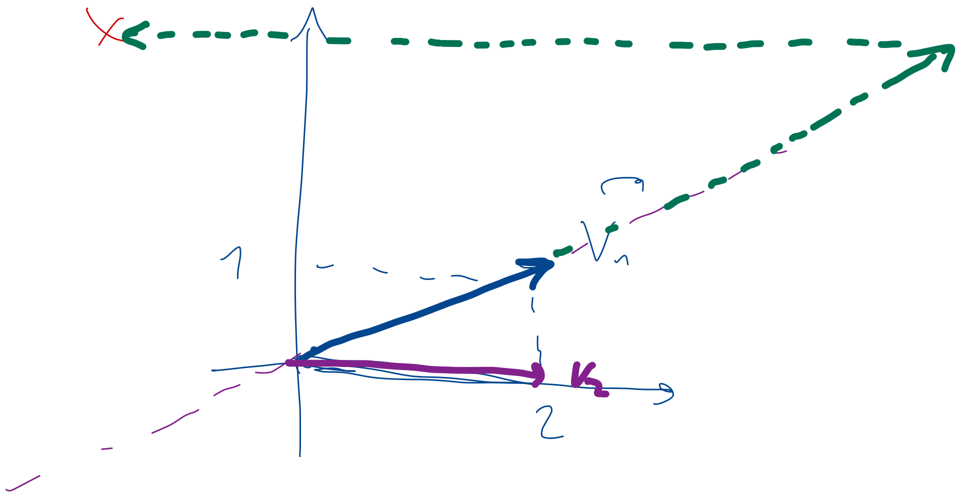
$\text{span}\{\vec{v}_1\} = c\vec{v}_1$ for any $c \in \mathbb{R}$

$\vec{v}_2 = -\vec{v}_1$

$\text{span}\{\vec{v}_1, \vec{v}_2\} = c\vec{v}_1$

Example

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



span is now all of $\mathbb{R}^2$ (2D plane)

Exercise

Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and let $\vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Is $\vec{y} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2)$?

If so, write $\vec{y}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

not parallel

$\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ $\quad -2 \cdot \vec{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$ $\quad \vec{v}_1 \nparallel \vec{v}_2$

$\text{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$ any vector in $\mathbb{R}^2$ can be written as lin. comb. of $\vec{v}_1, \vec{v}_2$

$w_1 \vec{v}_1 + w_2 \vec{v}_2 = \vec{y}$

$\begin{bmatrix} 2w_1 \\ -3w_1 \end{bmatrix} + \begin{bmatrix} -w_2 \\ 4w_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} 2w_1 - w_2 = 9 \\ -3w_1 + 4w_2 = 1 \end{matrix} \Rightarrow$ solve for w_1, w_2

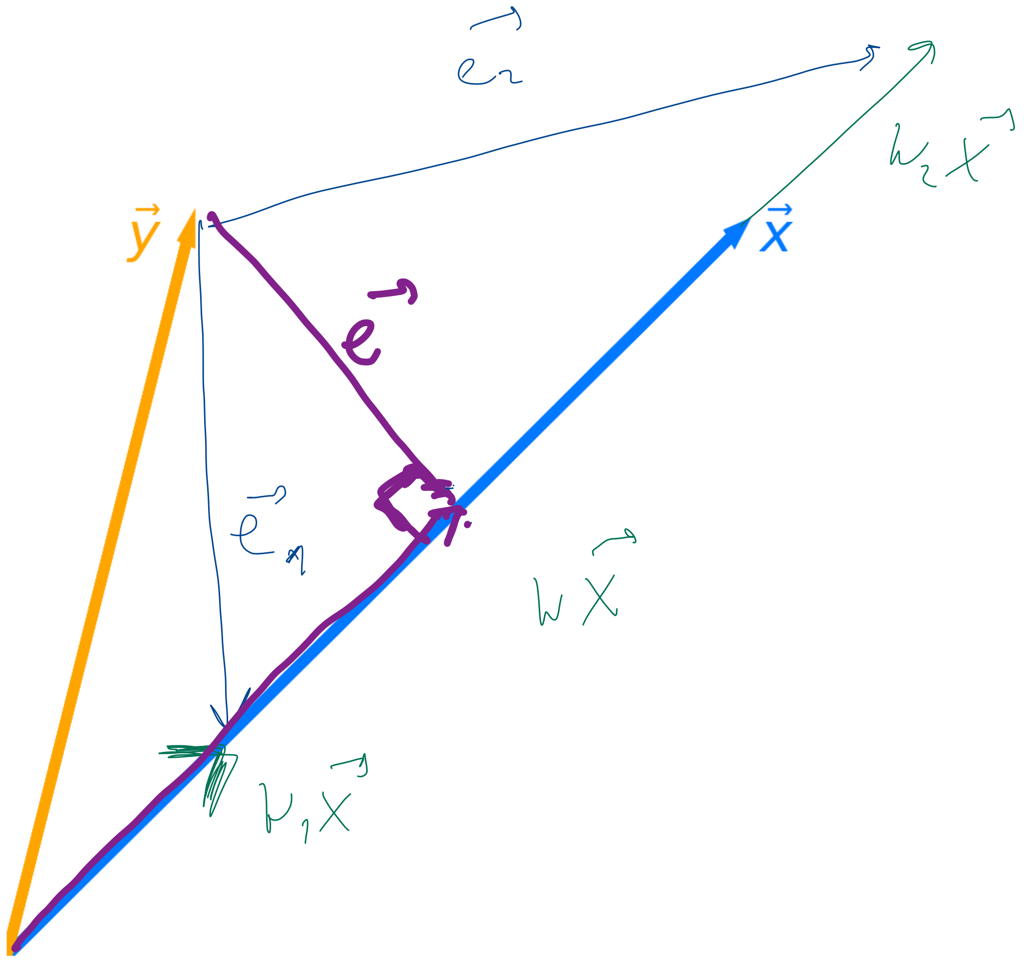
Projecting onto a single vector

- Let $\vec{x}$ and $\vec{y}$ be two vectors in $\mathbb{R}^n$.
- The span of $\vec{x}$ is the set of all vectors of the form:

$$w\vec{x}$$

  where $w \in \mathbb{R}$ is a scalar.

- **Question**: What vector in $\text{span}(\vec{x})$ is closest to $\vec{y}$?
- The vector in $\text{span}(\vec{x})$ that is closest to $\vec{y}$ is the orthogonal **projection of $\vec{y}$ onto $\text{span}(\vec{x})$**.



Projection error

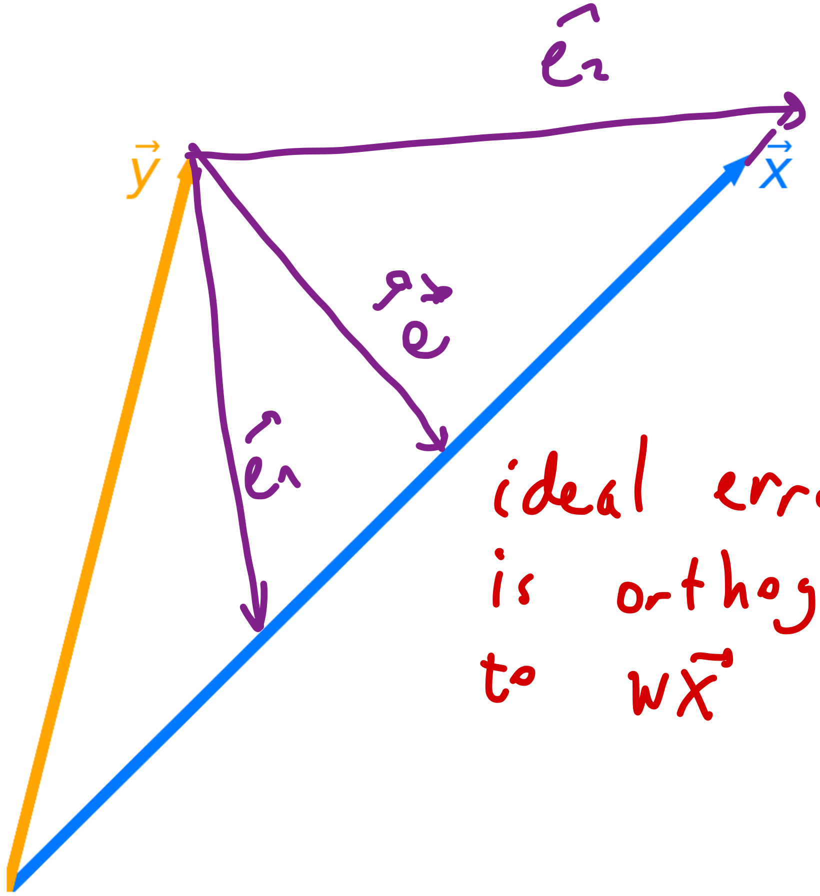
- Let $\vec{e} = \vec{y} - w\vec{x}$ be the **projection error**: that is, the vector that connects $\vec{y}$ to $\text{span}(\vec{x})$.
- **Goal**: **Find the w that makes $\vec{e}$ as short as possible**.
  - That is, minimize:

$$\min \quad \|\vec{e}\|$$

  - Equivalently, minimize:

$$\|\vec{y} - w\vec{x}\|$$

- **Idea**: To make $\vec{e}$ has short as possible, it should be **orthogonal to $w\vec{x}$**.



ideal error is orthogonal to $w\vec{x}$

Minimizing projection error

- **Goal: Find the w that makes $\vec{e} = \vec{y} - w\vec{x}$ as short as possible.**
- Now we know that to minimize $\|\vec{e}\|$, $\vec{e}$ must be orthogonal to $w\vec{x}$.
- Given this fact, how can we solve for w?

$$\min \|\vec{e}\| \Rightarrow \min \|\vec{y} - w\vec{x}\|$$

$$\vec{e} \perp w\vec{x} \Rightarrow \vec{e} \cdot w\vec{x} = 0$$

$$(\vec{y} - w\vec{x}) \cdot (w\vec{x}) = 0 \quad / \text{divide by } w$$

$$\vec{x} \cdot (\vec{y} - w\vec{x}) = 0$$

$$\vec{x} \cdot \vec{y} - \vec{x} \cdot (w\vec{x}) =$$

$$\vec{x} \cdot \vec{y} - w(\vec{x} \cdot \vec{x}) = 0$$

$$w\|\vec{x}\|^2 = \vec{x} \cdot \vec{y}$$

$$\boxed{w^* = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}}$$

optimal proj so that $\vec{e}$ is orthogonal to $\vec{x}$

Exercise

Let $\vec{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$.

What is the orthogonal projection of $\vec{a}$ onto $\text{span}(\vec{b})$?

Your answer should be of the form $w^*\vec{b}$, where w^* is a scalar.

Moving to multiple dimensions

- Let's now consider three vectors, $\vec{y}$, $\vec{x}^{(1)}$, and $\vec{x}^{(2)}$, all in $\mathbb{R}^n$.
- **Question**: What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to $\vec{y}$?
  - Vectors in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ are of the form $w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}$, where w_1, $w_2 \in \mathbb{R}$ are scalars.
- Before trying to answer, let's watch 🎥 **this animation that Jack, one of our tutors, made**.