DSC 40A

Theoretical Foundations of Data Science I

Foundations of Probability – Conditional Probability

Announcements

- Homework 5 released today
- Midterm grade report will be released today.

Agenda

- Multiplication rules and independence
- Conditional probability

Question

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!
If the direct link doesn't work, click the "Lecture Questions" link in the top right corner of dsc40a.com.

Multiplication rule and Independence

- The probability that events A and B both happen is

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$$

- $\mathbb{P}(B|A)$ means "the probability that B happens, given that A happened." It is a **conditional probability**.
  - More on this soon!
- If $\mathbb{P}(B|A) = \mathbb{P}(B)$, we say A and B are **independent**.
  - Intuitively, A and B are independent if knowing that A happened gives you no additional information about event B, and vice versa.
  - For two independent events, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Practice Problems

Example 2. A die is rolled 3 times. What is the probability that the face 1 never appears in any of the rolls?

$S = \{$ all trios of rolls $\}$ Ex: 123 656 425

$E = \{1$ did not appear in any roll$\}$

$P\{$not rolling $1\} = 1 - \frac{1}{6} = \frac{5}{6}$ complement rule

rolls are independent

$$P\{1 \text{ not in any roll}\} = P(\underset{\text{in roll 1}}{\overset{1 \text{ not}}{A}}) \cdot P(\underset{\text{in roll 2}}{\overset{1 \text{ not}}{B}}) \cdot P(\underset{\text{in roll 3}}{\overset{1 \text{ not}}{C}}) = \left(\frac{5}{6}\right)^3$$

$\frac{5}{6}$ $\frac{5}{6}$ $\frac{5}{6}$

$$P(A \text{ and } B \text{ and } C) = P(A)\,P(B|A) \cdot P(C|B \text{ and } A) \overset{\text{independence}}{=} P(A)P(B)P(C)$$

Practice Problems

Example 3. A die is rolled n times. What is the chance that only faces 2, 4 or 6 appear?

S for single roll $= \{1, 2, 3, 4, 5, 6\}$

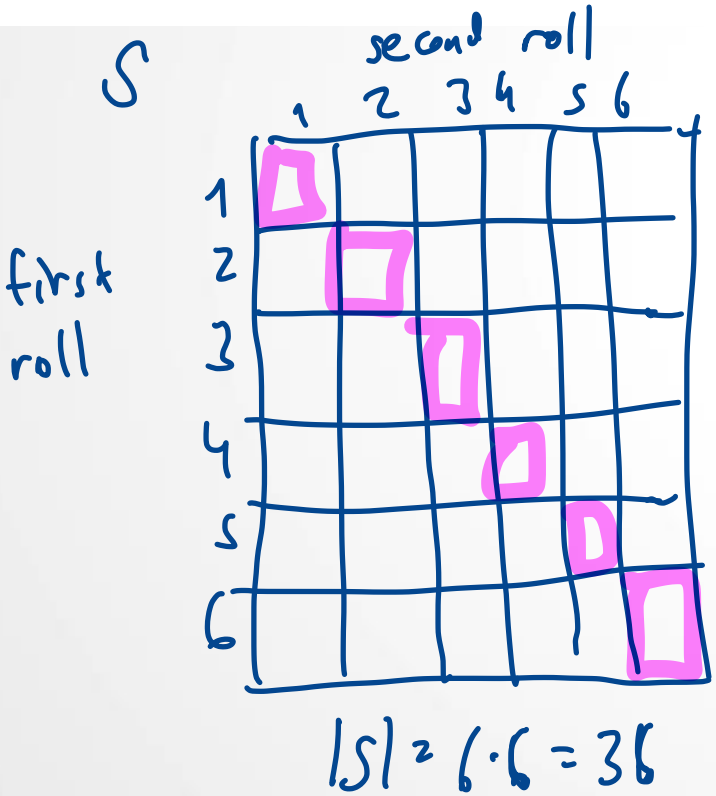
$E = \{2, 4, 6\}$

$P(E) = \frac{3}{6} = \frac{1}{2}$

Rolls are independent: $\underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n \text{ times}} = \left(\frac{1}{2}\right)^n$

Practice Problems

Example 4. A die is rolled two times. What is the probability that the two rolls had different faces?



Solution 1

$E = \{\text{two different faces}\}$

$\bar{E} = \{\text{both rolls had same face}\}$

$= \{11, 22, 33, 44, 55, 66\}$

complement

$P(E) = 1 - P(\bar{E}) =$

$1 - \frac{6}{36} = \frac{5}{6}$

Solution 2

$S = \{\text{second roll}\}$

$= \{1, 2, 3, 4, 5, 6\}$

$E = \{\text{second roll is different from first}\}$

$P(E) = \frac{5}{6}$

Conditional probabilities

Probability of an event may **change** if have additional information about outcomes.

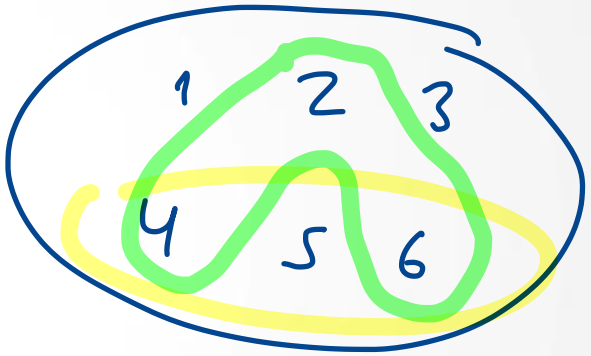
Rolling a die

What is the probability of rolling a number >3 ?

$E = \{4, 5, 6\}$

$$P(E) = \sum_{s \in E} p(s) = \sum_{s \in \{4,5,6\}} \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$S =$



$P(E \mid F) \overset{?}{=} P(E)$

Extra info: result was even

$F = \{2, 4, 6\}$

Conditional probabilities

Probability of an event may **change** if have additional information about outcomes.

Suppose E and F are events, and P(F) > 0. Then,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$E = \{4, 5, 6\}$

$F = \{2, 4, 6\}$

$P(F) = \frac{\#F}{\#S} = \frac{3}{6} = \frac{1}{2}$

$P(E|F) = P(E)$

i.e.,

$$P(E \cap F) = P(E|F)P(F)$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \left(\frac{\#\text{ of outcomes in } E \text{ and } F}{\#\text{ of outcomes in } S}\right) \cdot \frac{1}{P(F)} = \frac{\frac{2}{6}}{\frac{1}{2}} = \frac{4}{6} = \frac{2}{3} > \frac{1}{2}$$

Conditional probabilities

Are these probabilities equal?

Suppose a family has two pets. Assume that it is **equally likely** that each pet is a dog or a cat. Consider the following two probabilities:

- The probability that both pets are dogs given that **the oldest is a dog.**
- The probability that both pets are dogs given that **at least one of them is a dog.**

What do you think?

A. they are equal 70 %.
B. they are not equal 30%.

Conditional probabilities

Are these probabilities equal?

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$S = \{dd, cc, dc, cd\}$ — $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

$E = \{dd\}$ $\quad P(E) = \frac{1}{4}$

$E \subset F$

$F = \{dd, dc\}$ $\quad P(F) = \frac{2}{4} = \frac{1}{2} > 0$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{2}{4} = \frac{1}{2}$$

$E \subset F$

Conditional probabilities

Are these probabilities equal?

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$S = \{dd, cc, dc, cd\}$

$E = \{dd\}$ $\quad P(E) = \frac{1}{4}$

$F = \{dd, dc, cd\}$ $\quad P(F) = \frac{3}{4} > 0$

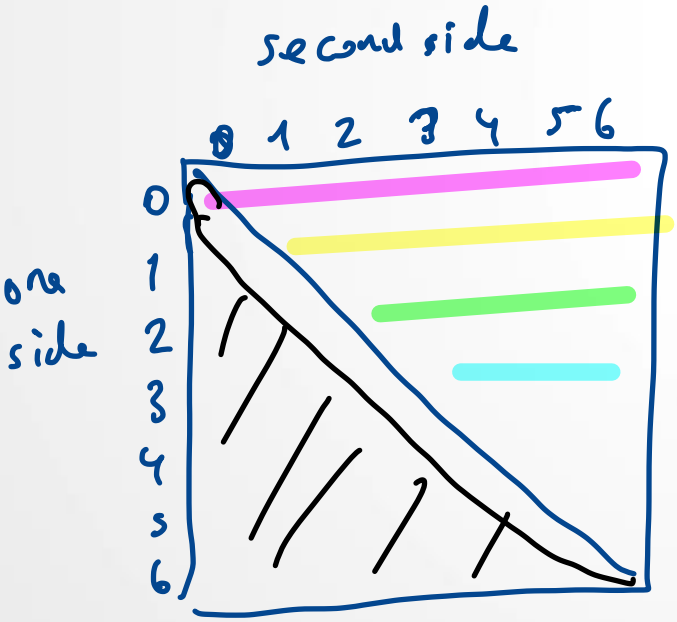
$E \subset F \Rightarrow E \cap F = E$

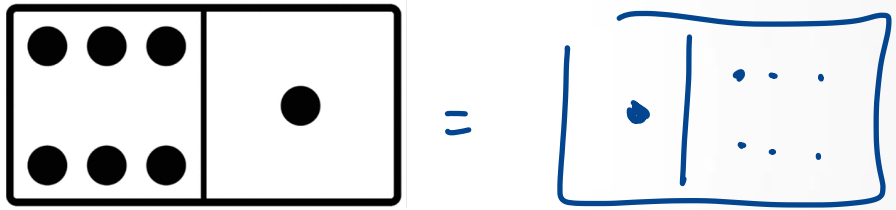
$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} < \frac{1}{2}$$

Dominoes

In a set of dominos, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.





$7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$ tiles

Five Thirty Eight Riddler Express: https://fivethirtyeight.com/features/can-you-eat-an-apple-like-a-toddler/

Dominoes

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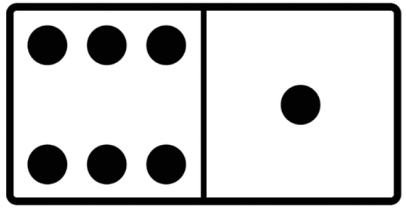

Question 1: What is the probability of drawing a “double” from a set of dominoes — that is, a tile with the same number on both sides?

Question 2: Now you pick a random tile from the set and uncover only one side, revealing that it has six dots. What’s the probability that this tile is a double, with six on both sides?

Question 3: Now your friend picks a random tile from the set and looks at it. You ask if they have a six, and they answer yes. What is the probability that your friend’s tile is a double, with six on both sides?

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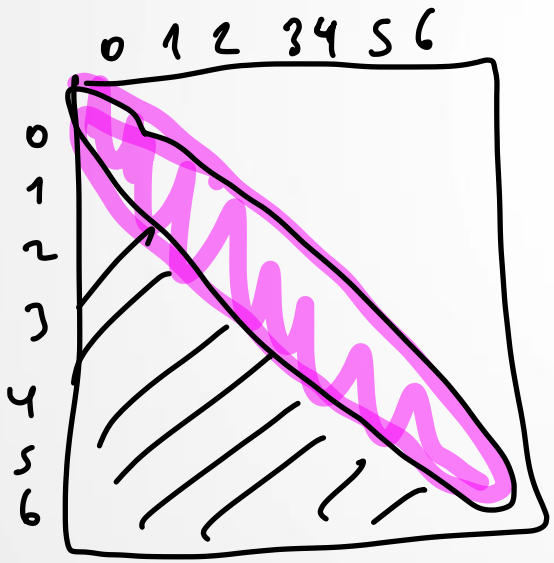
Dominoes

Question 1: What is the probability of drawing a “double” from a set of dominoes — that is, a tile with the same number on both sides?



$S = \{\text{all 28 tiles}\}$

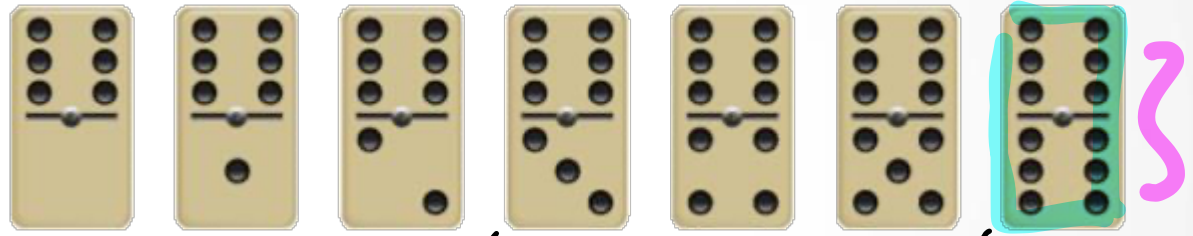
$E = \{00, 11, 22, 33, 44, 55, 66\}$

$$P(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{7}{28} = \frac{1}{4}$$

Five Thirty Eight Riddler Express: https://fivethirtyeight.com/features/can-you-eat-an-apple-like-a-toddler/

Dominoes

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?

$F =$ 

$S = \{\text{all 28 tiles}\}$

$E = \{66\}$

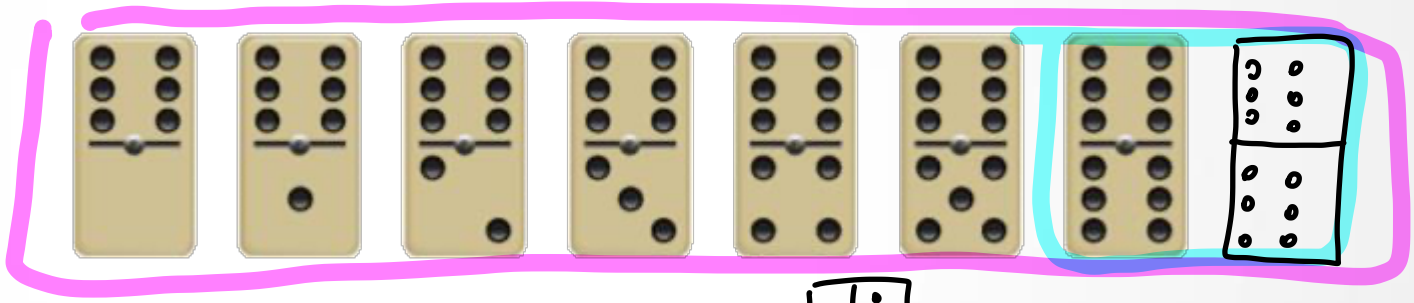
$P(F) = \frac{\#\text{outcomes in } F}{\#\text{outcomes in } S} = \frac{7}{28}$

$P(E|F) = \frac{P(E \cap F)}{P(F)} \overset{ECF}{=} \frac{P(E)}{P(F)} = \frac{\frac{1}{28}}{\frac{7}{28}} = \frac{1}{7}$

$P(E \cap F) = P(E) = \frac{1}{28}$

Dominoes

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has 6 dots. What is the probability that this tile is a double, with 6 on both sides?



S = 28 · 2 possible domino layout = 56

$\Rightarrow$ both sides are the same  7 · 2 = 14

E = both sides are the same with six on both sides = 2

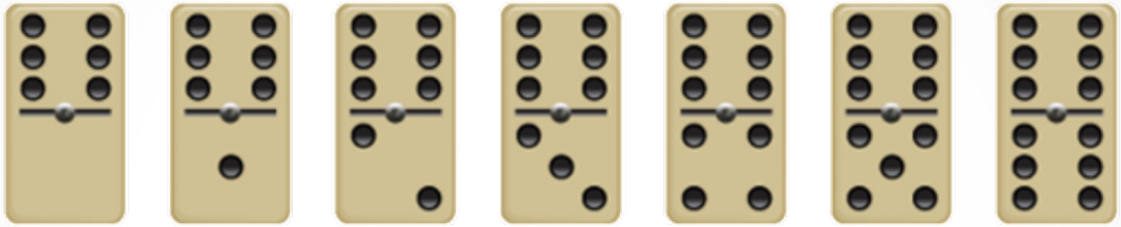
F = layout with 6 on one side = 8 outcomes

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/56}{8/56} = 1/4 > 1/7$$

Five Thirty Eight Riddler Express: https://fivethirtyeight.com/features/can-you-eat-an-apple-like-a-toddler/

Dominoes

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has 6 dots. What is the probability that this tile is a double, with 6 on both sides?



Try it out in code!

Five Thirty Eight Riddler Express Solution: https://fivethirtyeight.com/features/can-you-find-the-best-dungeons-dragons-strategy/