DSC40B: Theoretical Foundations of Data Science II

Lecture 7: *The Median, order statistics, QuickSelect and QuickSort*

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Previously

- Sorting an array
- (Binary) search in a sorted array

- Today:
  - What if, without sorting, we would like to select a specific number with a certain rank in the array
  - For example, how to find the median of an unsorted array of numbers quickly?

Part A:
Order statistics and simple examples

Order statistics

→ 20, 10, 21, 5, 8.

3rd order statistic : 10
rank-3 element.

- Given a set of n numbers
  - The **kth order statistics** is the kth smallest number in this collection
    - We also say that this number has *rank k* in the input.

5th ~ : 21.

- Examples:
  - 1^{st} order statistics: minimum
  - nth order statistics: maximum
  - $\lceil \frac{n}{2} \rceil$-th order statistics: median
  - $\lceil \frac{pn}{100} \rceil$-th order statistics: p-th percentile

Select problem

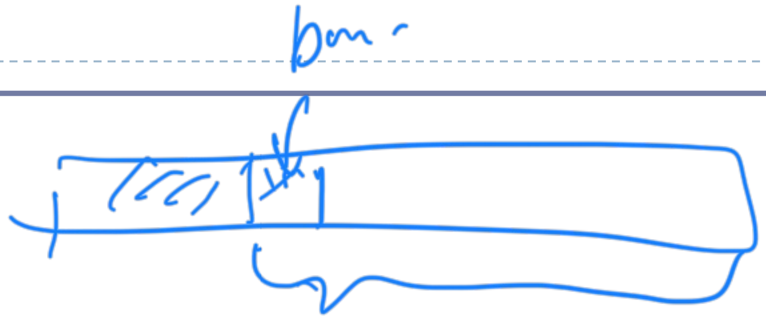
- Input: given n numbers stored in an array A, and an order (rank) $k \in [1, n]$
- Output: return the k-th order statistics of A

- Special cases:
  - $k = 1$? $k = n$?
  - But how about for general k, including finding the median of A?

Simple approaches

- Approach 1:
  - Modifying selection sort
  - Stops when find the k-th order statistics

Algorithm selection_sort



```
def selection_sort(A):
    n = len(A)
    if n <= 1:
        return
    for barrier_id in range(n-1):
        # find index of min in A[start:]
        min_id = find_minimum(A, start=barrier_id)
        #swap
        A[barrier_id], A[min_id] = (
            A[min_id], A[barrier_id]
        )
```

Algorithm selection_kthOS

```
def selection_kthOS(A, k):
    n = len(A)
    if n < k:
        return Error
    for barrier_id in range(k):
        # find index of min in A[start:]
        min_id = find_minimum(A, start=barrier_id)
        #swap
        A[barrier_id], A[min_id] = (
            A[min_id], A[barrier_id]
        )
    return A[k-1]
```

$= n + (n-1) + \ldots + (n-k+1)$

$= \Theta(nk)$

$\Theta(nk)$

Simple approaches

- Approach 1:
  - Modifying selection sort
  - Stops when find the k-th order statistics
  - Time complexity
    - $\Theta(kn)$
- Approach 2:
  - First sort array A
  - Return $A[k]$
  - Time complexity
    - Same as sorting, which is $\Theta(n \lg n)$

Can we do better than sorting (namely $\Theta(n \lg n)$ time)?

Part B:
Can we do better than sorting?
First try of *QuickSelect*

I will use pseudo-code in what follows.
As convention: array index starts from 0.

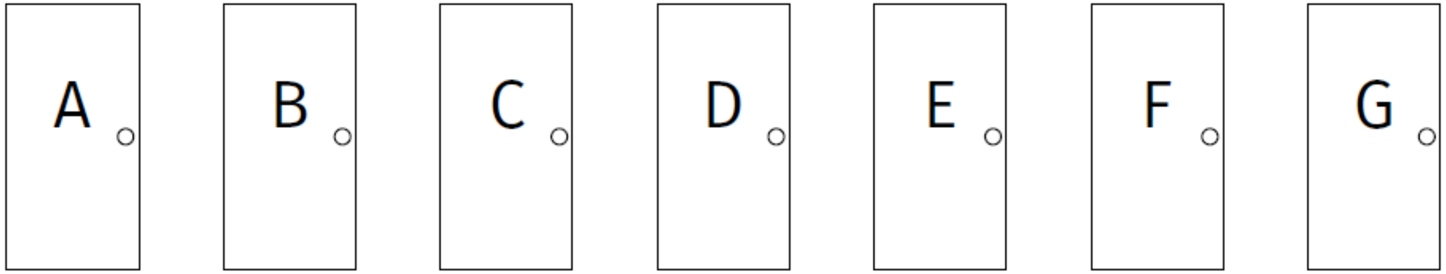
Select problem

- Input: given n numbers stored in an array A, and an order $k \in [1, n]$
- Output: return the k-th order statistics of A

- Intuition:
  - In Sorting, we essentially figure out the relative orders among all elements
    - There is much redundancy; for example, if two numbers both have higher order than the target order k, then intuitively, we don't care about spending time to figure out their relative order.
    - So intuitively, we should be able to do better than sorting.
  - How to leverage this thought?

An example

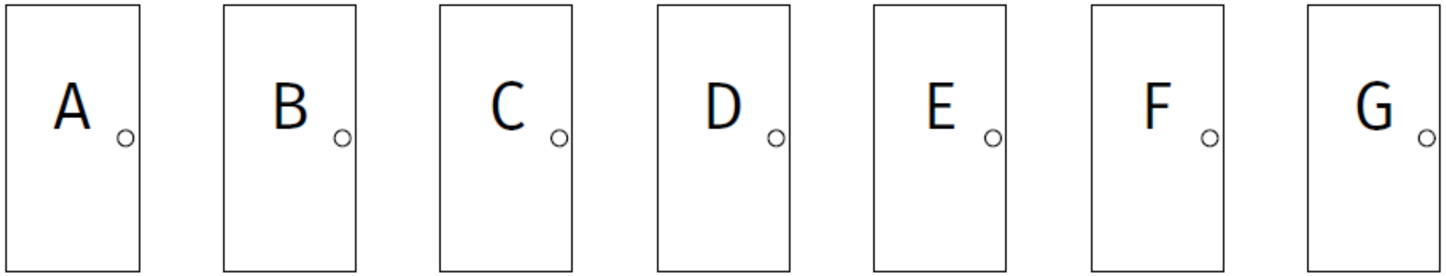
- Given n doors, need to find the largest number behind the door
- Each time we open a door, we have an oracle to tell us
  - which doors are smaller, and
  - which doors are bigger

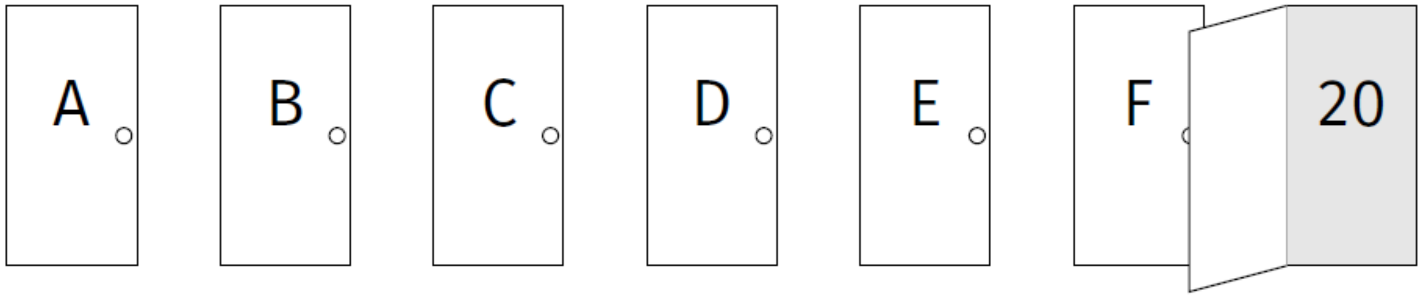


An example

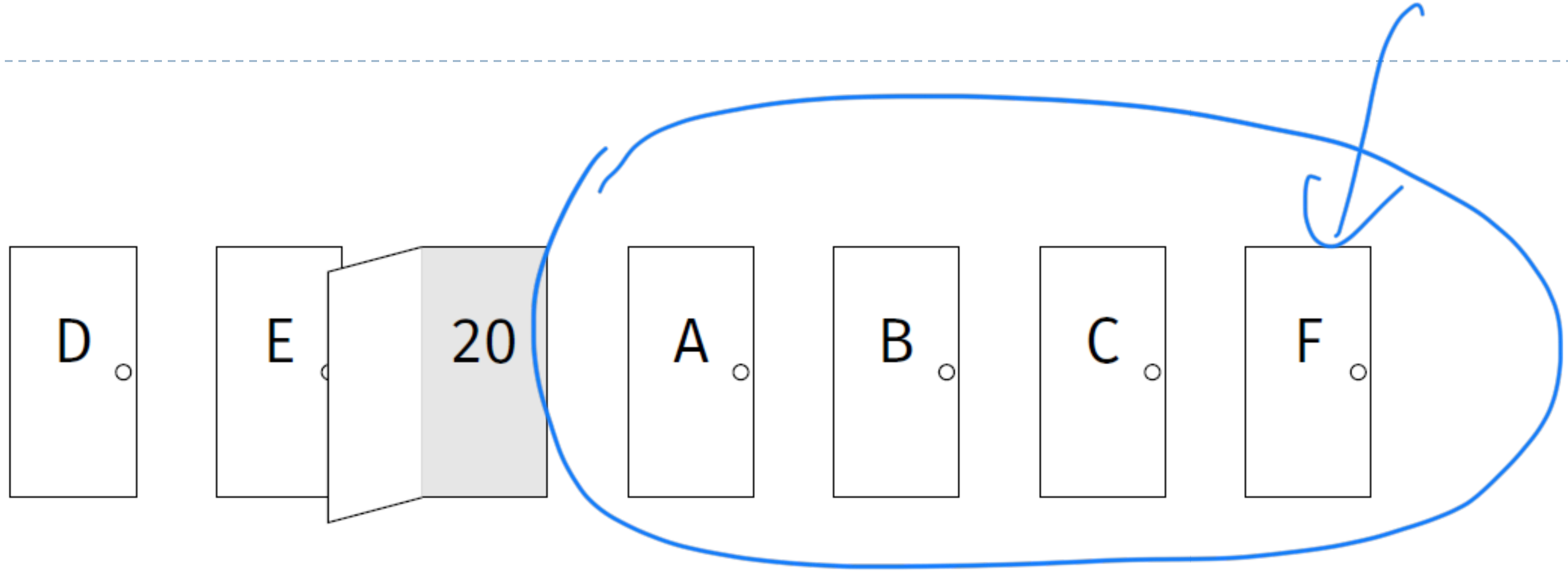
- Given n doors, need to find the largest number behind the door
- Each time we open a door, we have an oracle to tell us
  - which doors are smaller, and
  - which doors are bigger

Call this a partition operation

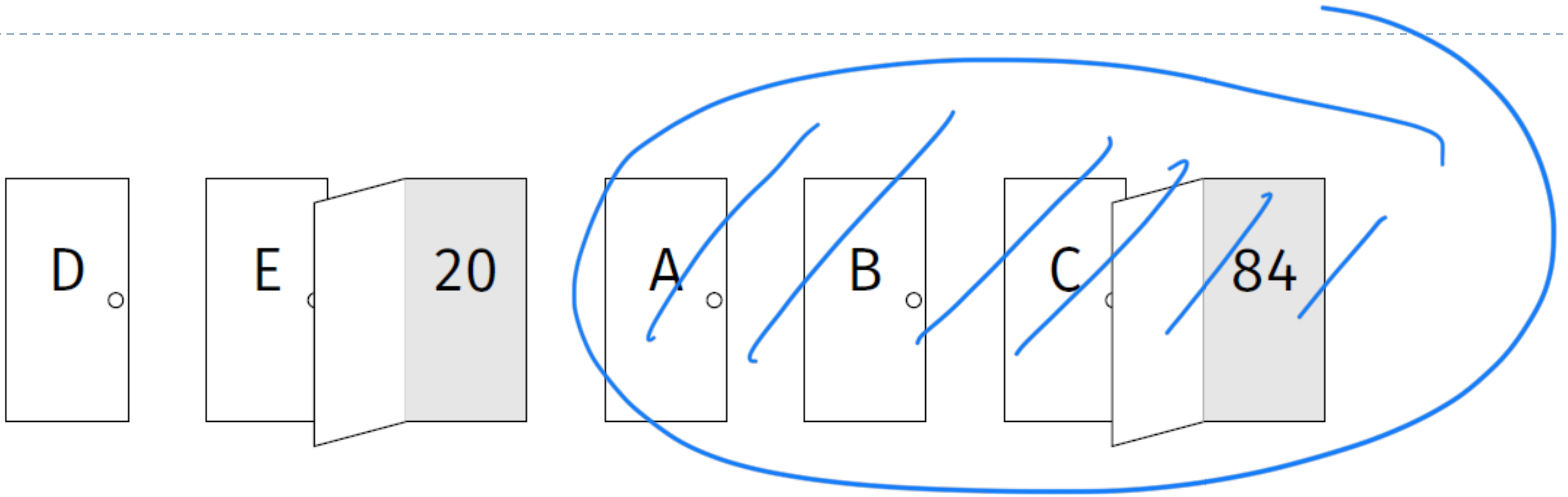




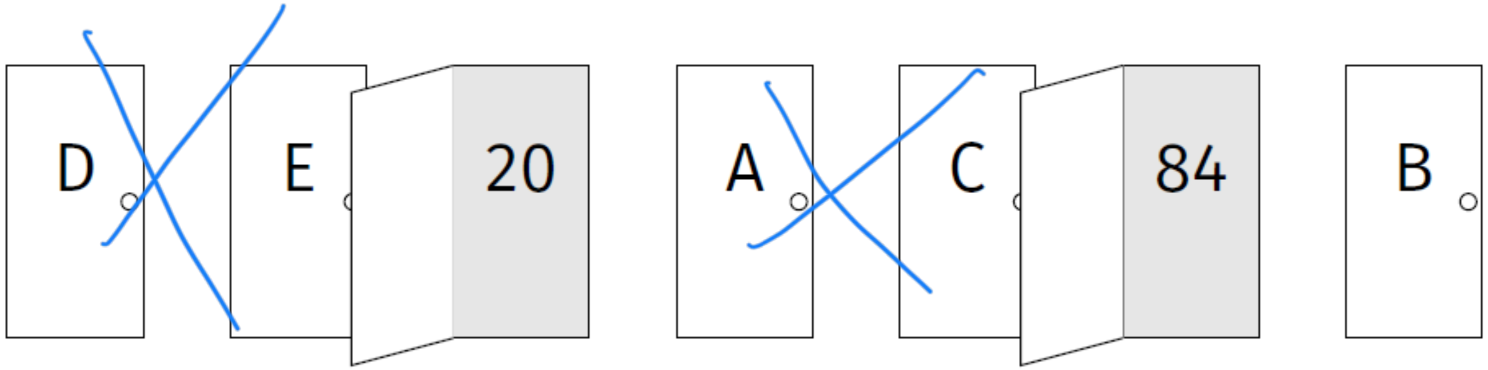
we open the last door



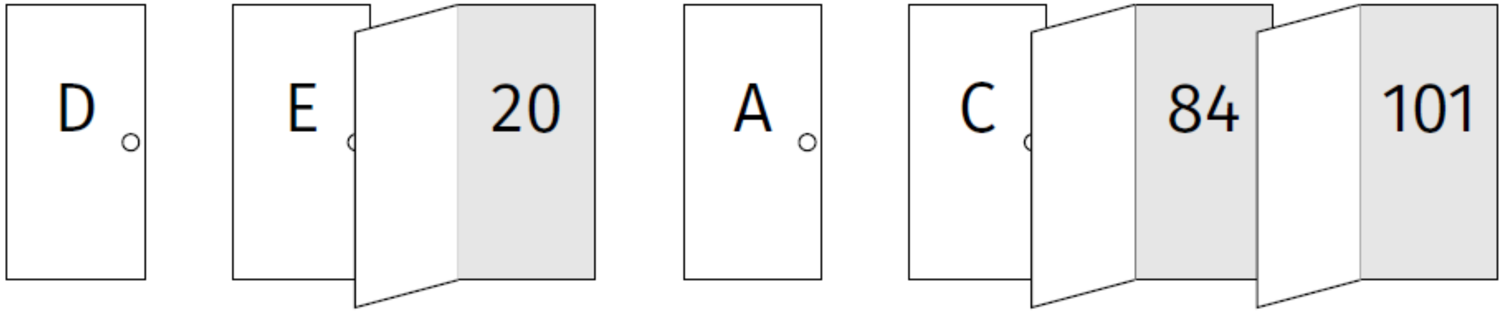
after partition



repeat in the right portion:
open the last door of this subarry



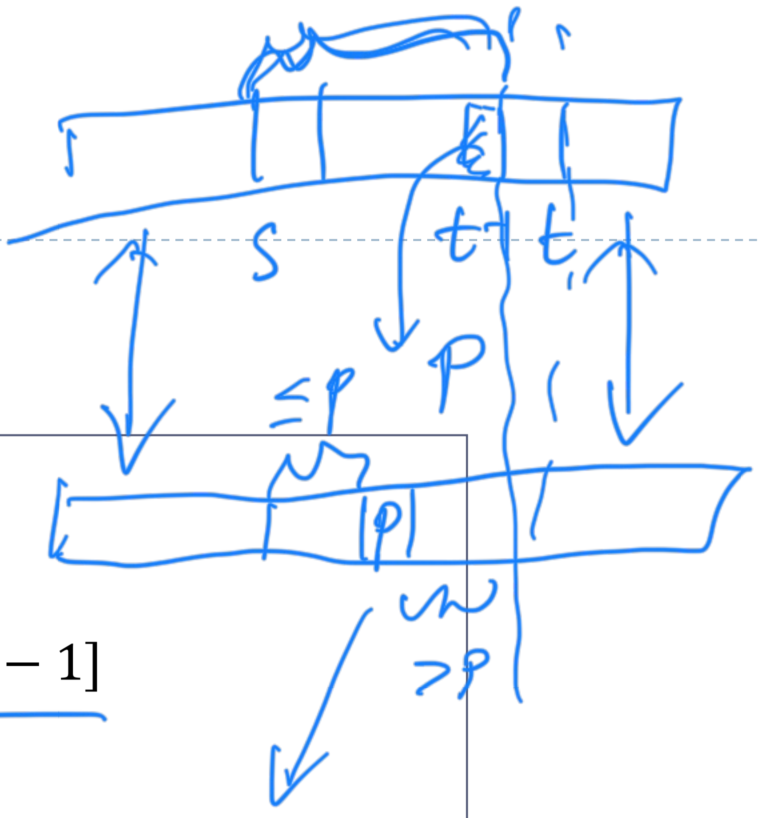
repeat in the right portion:
after partition in this subarray



again, go to the right portion:
only 1 entry left: must be the largest,
and we return

Generalizing the idea?

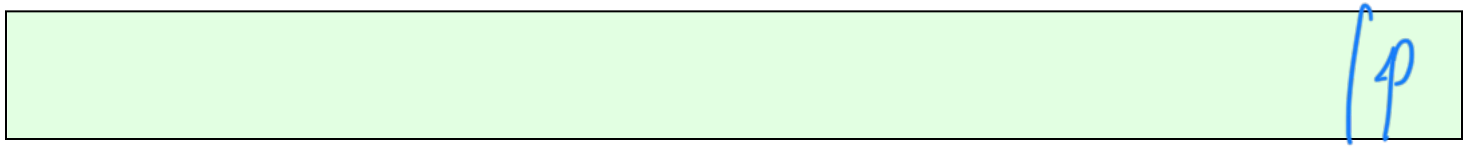
- Assume that we are given the Partition procedure:



- **Partition (A, s, t)**
  - Input:
    - Given an array A and consider sub-array $A[s, \dots\ t-1]$
    - $A[t-1]$ will be used as the pivot $p = A[t-1]$
  - Output:
    - Rearrange elements in A where p is now in $A[m]$ such that
      - □ all elements $\leq p$ are to its left
      - □ all elements $> p$ are to its right
    - Return the new position m of the pivot p

Intuition of QuickSelect

- Suppose we are already given the Partition procedure.
- QuickSelect$(A, 0, n, k)$
  - $m =$ Partition$(A, 0, n)$
  - Note: the order of the pivot $= m + 1$



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- Suppose we are already given the Partition procedure.
- QuickSelect$(A, 0, n, k)$
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$p = A[m]$: pivot

If $k = m+1$

$k < m+1$

Intuition of QuickSelect

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- QuickSelect$(A, 0, n, k)$
  - m = Partition$(A, 0, n)$
  - Note: the order of the pivot = $m + 1$

$p = A[m]$: pivot

Case 1: $k = m+1$

return $A[m]$

Intuition of QuickSelect

- Suppose we are already given the Partition procedure.
- QuickSelect$(A, 0, n, k)$
  - m = Partition$(A, 0, n)$
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$p = A[m]$: pivot

Case 2: $k < m+1$

Intuition of QuickSelect

- Suppose we are already given the Partition procedure.
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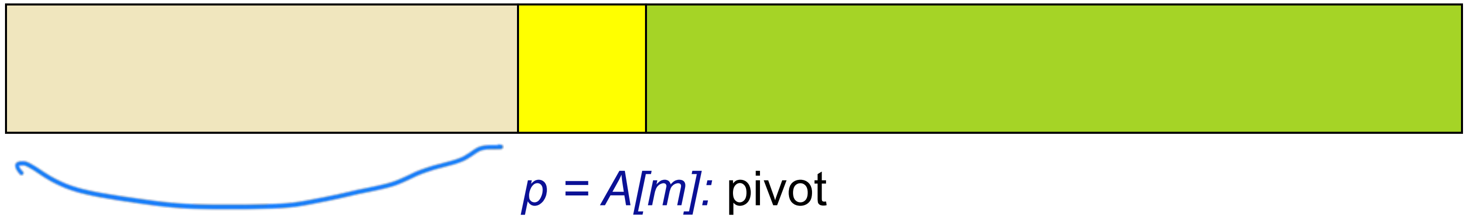
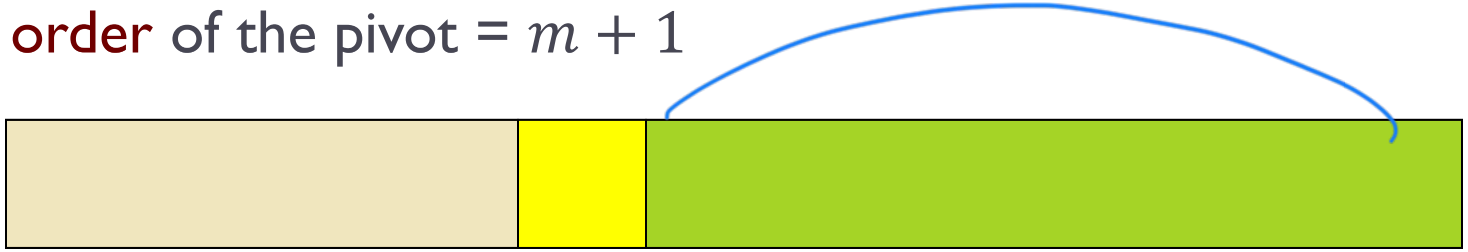


Case 2: $k < m+1$

return QuickSelect $(A, 0, m, k)$

Intuition of QuickSelect

- Imagine we are given Partition procedure.
- QuickSelect$(A, 0, n, k)$
  - m = Partition$(A, 0, n)$
  - Note: the order of the pivot = $m + 1$



p = A[m]: pivot

Case 3: $k > m+1$

return QuickSelect (A, $m+1$, n, k)

Pseudo-code for QuickSelect

```
QuickSelect ( A, s, t, k )
 /* select the order k element in A from subarray A[s,..t-1] */
   if (k < s or k ≥ t or s ≥ t) return None;
   m = Partition ( A, s, t );
   pivot_order = m+1 ;
   if ( pivot_order = k)  return A[m];
   if ( pivot_order > k )
       return QuickSelect ( A, s, m, k );
   else  return QuickSelect ( A, m+1, t, k );
```

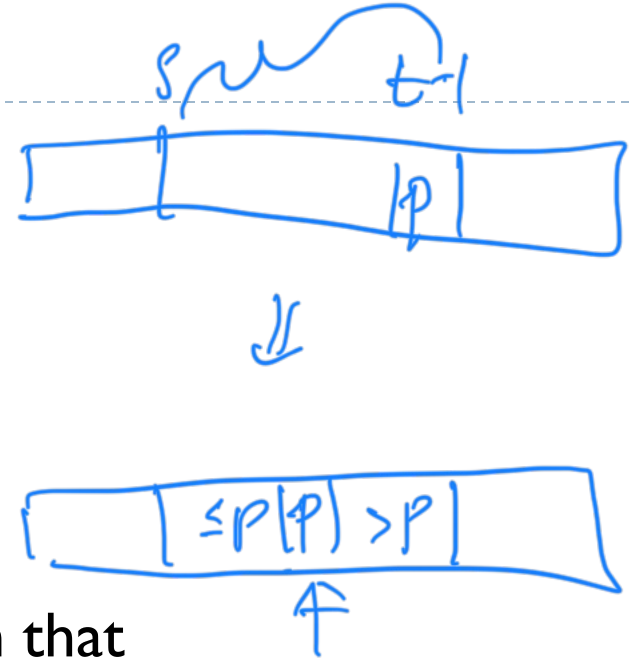
At the top level, we call QuickSelect(A, 0, n, k)

Example

- $A = [13, 2, 5, 9, 4, 6]$
- Goal: find 2nd order statistics in A; i.e, $k = 2$

Part C:
Partition procedure

Partition procedure



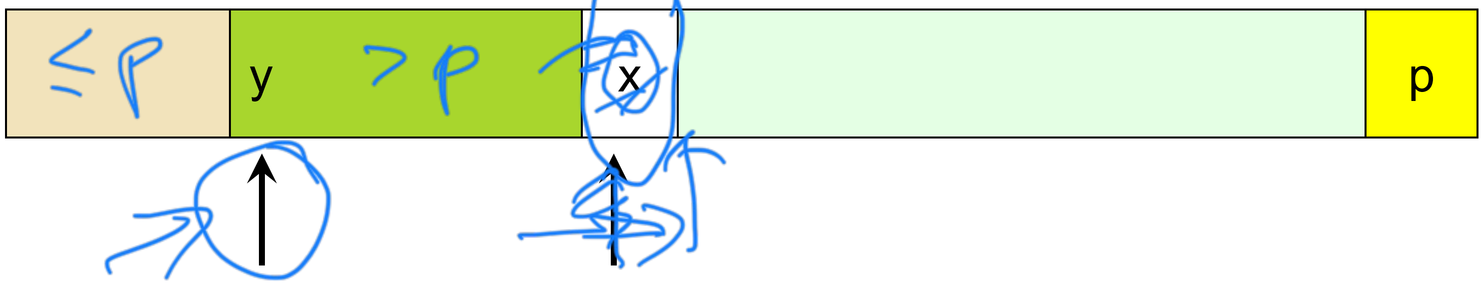
- Partition (A, s, t)
  - Input:
    - Given an array A and consider sub-array $A[s, \ldots\ t-1]$
    - $A[t-1]$ will be used as the pivot $p = \ A[t-1]$
  - Output:
    - Rearrange elements in A where p is now in $A[m]$ such that
      - ☐ all elements $\leq p$ are to its left
      - ☐ all elements $> p$ are to its right
    - Return the new position m of the pivot p

Partition(A, s, t)

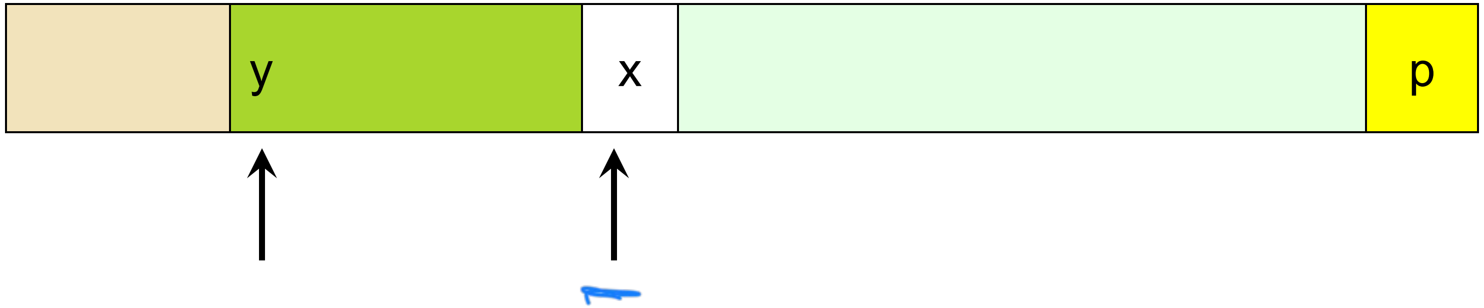
Plan: take *A[t-1]* as pivot

return *m*

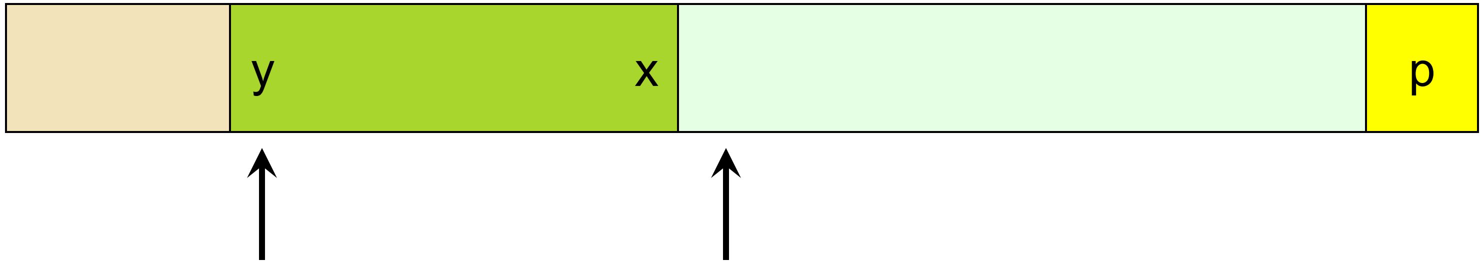
In-place partition !
i.e, we use the same input array,
and only need constant number of auxiliary memory



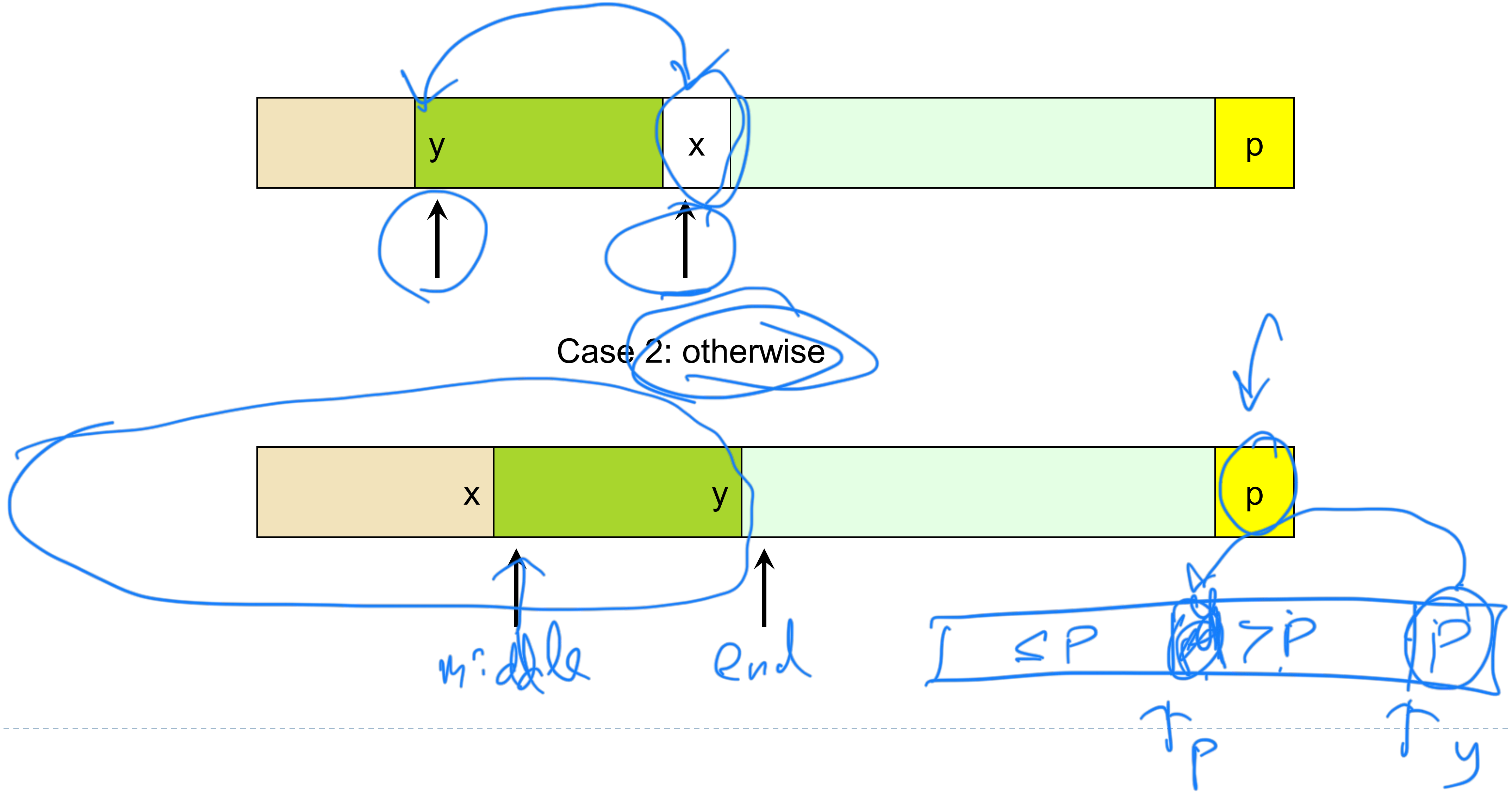
Partition(A, s, t)



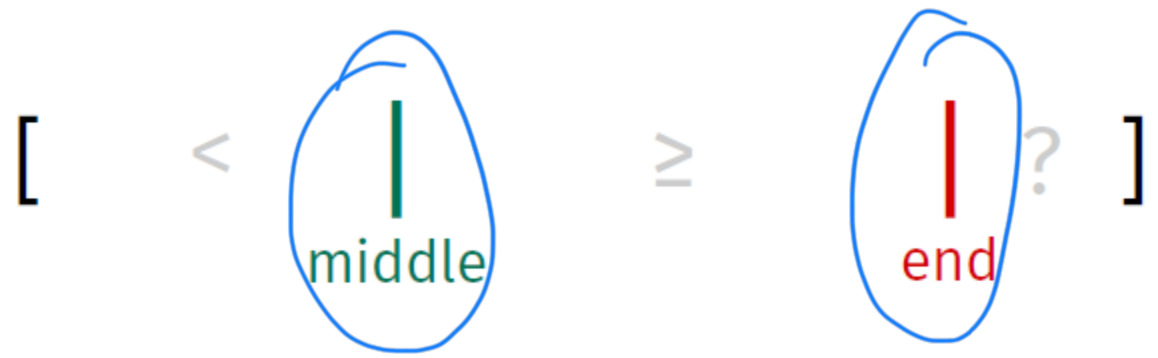
Case 1: x > p



Partition(A, s, t)



- Maintain two pointers:
  - "middle" barrier (variable ℓ in code):
    - separates numbers $\leq p$ from those $> p$
    - points to the first number $> p$ so far
  - "end" barrier (variable r in code):
    - separates what's already processed from un-processed
    - points to the first unprocessed number



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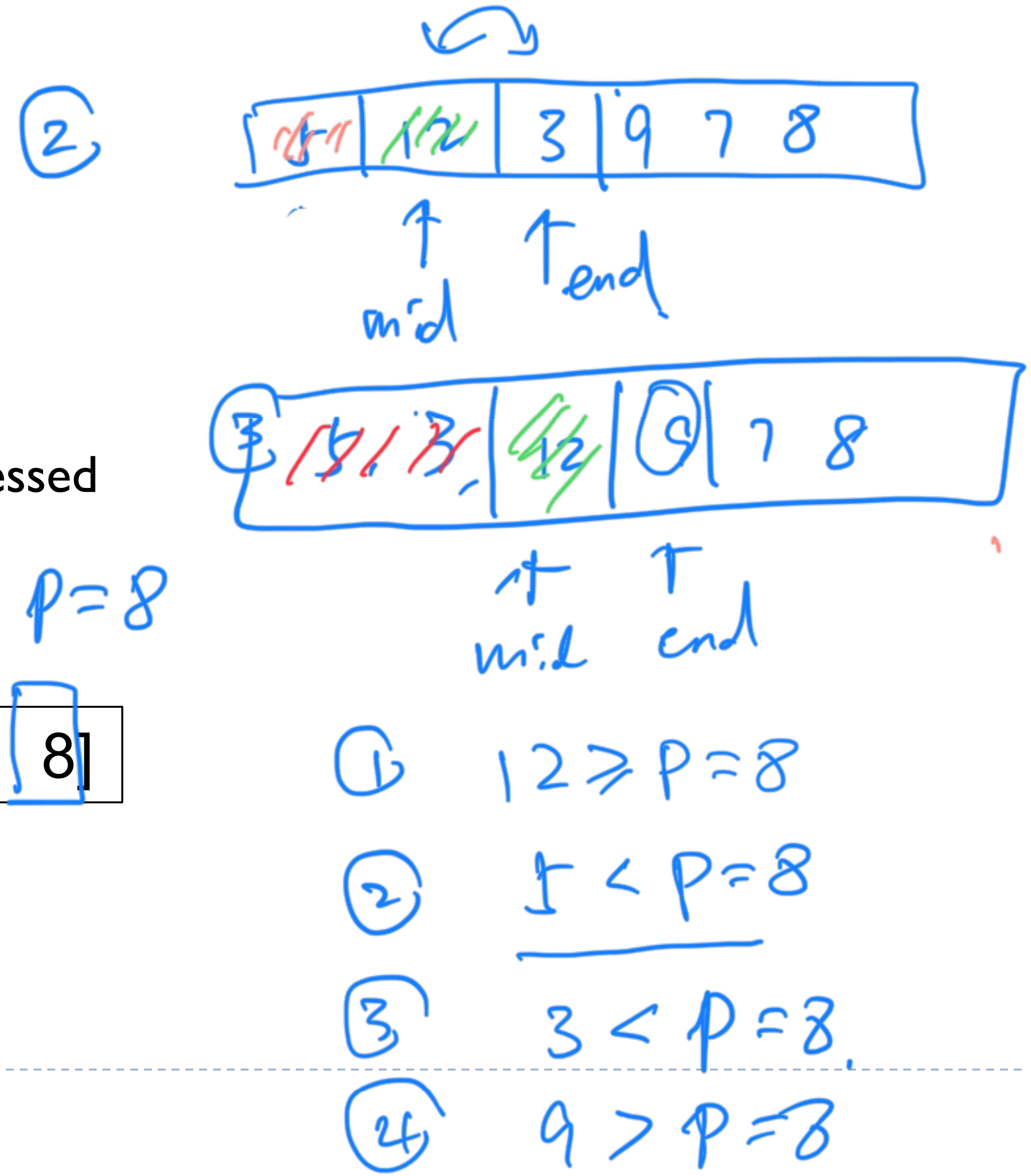
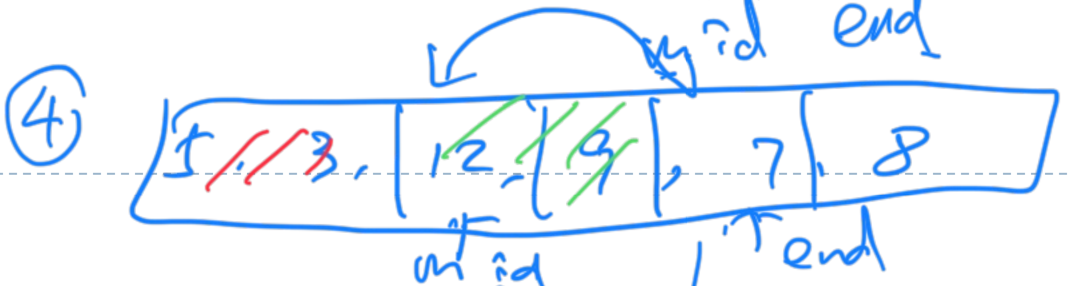
- Example: [12, 5, 3, 9, 7, 8]





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[12, 5, 3, 9, 7, 8]

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Pseudo-code for Partition

```
Partition(A, s, t)
/* Partition the subarray A[s, ..., t − 1] using A[t − 1] as pivot.
/* ℓ: index for mid_barrier; and r: index for end_barrier.

ℓ = s;
for r = s to t − 2 do
    if A[r] ≤ p then
        exchange A[ℓ] with A[r];
        ℓ + +;
    end
end
exchange A[ℓ] with A[t − 1];
return (ℓ);
```

In-place!

Time complexity:

$\Theta(t - s)$

Pseudo-code for Partition

```
Partition(A, s, t)
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    end
end
exchange A[ℓ] with A[t − 1];
return (ℓ);
```

Part D:
Time complexity for QuickSelect and Randomized QuickSelect

Worst case complexity

```
QuickSelect ( A, s, t, k )
 /* select the order k element in A from subarray A[s,..t-1] */
    if (k < s or k ≥ t or s ≥ t) return None;
    m   = Partition ( A, s, t );
    pivot_order = m+1 ;
    if ( pivot_order = k)  return A[m];
    if ( pivot_order > k )
        return QuickSelect ( A, s, m, k );
    else  return QuickSelect ( A, m+1, t, k );
```

Worst case complexity

```
QuickSelect ( A, s, t, k )
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   if ( pivot_order = k)  return A[m];
   if ( pivot_order > k )
        return QuickSelect ( A, s, m, k );
   else  return QuickSelect ( A, m+1, t, k );
```

At the top level, we call QuickSelect(A, 0, n, k).

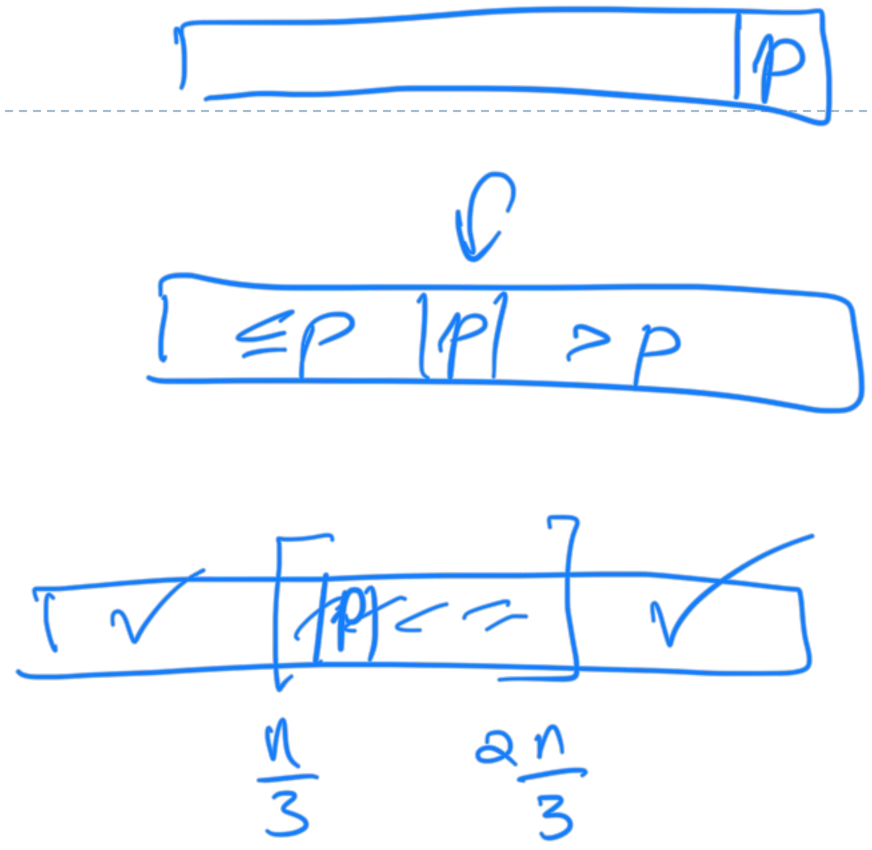
$T(n) = \max(T(r-1), T(n-r)) + cn$,where $r = m + 1$ is the pivot_order

$= \max(T(m), T(n-m-1)) + cn$

- $T(n) = \max\big(T(r-1), T(n-r)\big) + cn$
  - Depending on value of r, recursively. when $r = \frac{n}{2}$
- A lucky case: (Best choice of r for $T(n) = \max(T(r-1), T(n-r)) + cn$)
  - Each time we remove half of the numbers
    - we cannot do better, why?

$$T(n) = T(\frac{n}{2}) + cn$$
$$= \Theta(n)$$

- $T(n) = \max\big(T(r-1), T(n-r)\big) + cn$
  - Depending on value of r, recursively.



- A lucky case:
  - Each time we remove half of the numbers
    - we cannot do better, why?
  - $T(n) = T\left(\frac{n}{2}\right) + cn$

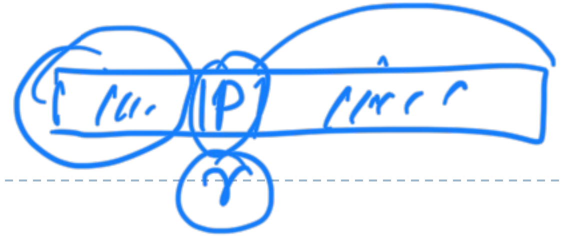
    $= \Theta(n)$

$$T(n) \leq T\left(\frac{2n}{3}\right) + cn$$

$$= \Theta(n)$$

$$\frac{n}{3} \leq rank(p) \leq \frac{2n}{3}$$

Best case when $r = \frac{n}{2}$.



- $T(n) = \max\big(T(r-1), T(n-r)\big) + cn$
  - Depending on value of m, recursively.

$r = 1$ or $r = n$.

- Worst case:
  - Each time we can only remove one number
    - say, the target order $k = n$, while $\mathrm{r}-1$ each time
  - $T(n) = T(n-1) + cn$

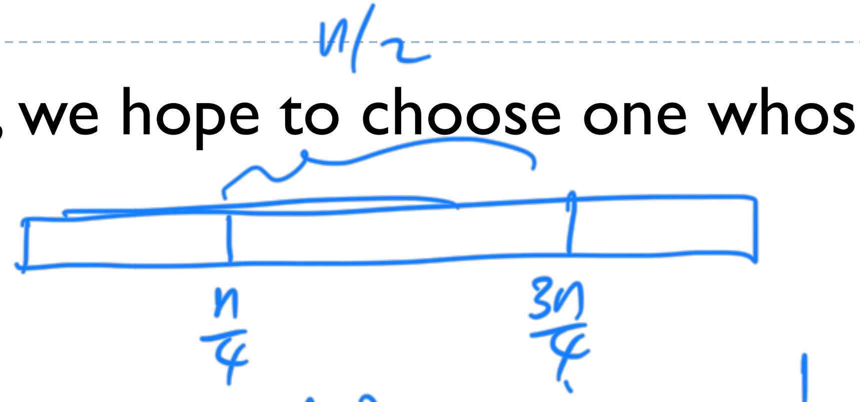
    $= \Theta(n^2)$

- How to ensure we mostly have "good cases"?
- Good split:
  - The pivot splits the current subarray in a balanced way (a constant fraction is on each side, say, the pivot_order r is such that $r \in [\frac{n}{4}, \frac{3n}{4}]$)
- Bad split:
  - Otherwise

- Roughly speaking, if we always have good splits, then we have that
  - $T(n) = \Theta(n)$
- In fact, this can be relaxed to that if we can have one good split every few (constant number of) splits on average

How to ensure that this happens?

- In other words, when we choose pivot, we hope to choose one whose rank (order) is around the middle
  - say, between $\frac{n}{4}$ to $\frac{3n}{4}$



$$\text{Prob}\left(\frac{n}{4} \le \text{rank}(p) \le \frac{3n}{4}\right) = \frac{1}{2}$$

$$T(n) = T\left(\frac{3n}{4}\right) + n$$

$$= T\left(\left(\tfrac{3}{4}\right)^2 n\right) + \frac{3n}{4} + n = T\left(\left(\tfrac{3}{4}\right)^3 n\right) + \left(\tfrac{3}{4}\right)^2 n + \left(\tfrac{3}{4}\right) n + n$$

$$\cdots = T\left(\left(\tfrac{3}{4}\right)^k n\right) + n + \frac{3n}{4} + \left(\tfrac{3}{4}\right)^2 n + \cdots + \left(\tfrac{3}{4}\right)^{k-1} n.$$

$$= T\left(\frac{n}{\left(\frac{4}{3}\right)^k}\right) + n\left(1 + \frac{3}{4} + \left(\tfrac{3}{4}\right)^2 + \cdots + \left(\tfrac{3}{4}\right)^{k-1}\right).$$

Step $k = \log_{\frac{4}{3}} n$

$\Theta(1)$

$= T(1) + n \cdot \Theta(1) = \Theta(n)$

- In other words, when we choose pivot, we hope to choose one whose rank (order) is around the middle
  - say, between $\frac{n}{4}$ to $\frac{3n}{4}$
- To guarantee that,
  - Pick a random number in A as the pivot!

- In other words, when we choose pivot, we hope to choose one whose rank (order) is around the middle
  - say, between $\frac{n}{4}$ to $\frac{3n}{4}$
- To guarantee that,
  - Pick a random number in A as the pivot!
- Why?
  - If we pick a random number $x \in A$
    - i.e, means that the probability of choose any one of the n numbers in A is $\frac{1}{n}$
    - Probability Pr[$rank(x) \ \in \left[\frac{n}{4}, \frac{3n}{4}\right]$] $= (\frac{3n}{4} - \frac{n}{4}) /\ n = 2/4 = 1/2$
    - Hence in expectation, every two times we will have a good split.

Rand-Select

```
Rand-Select ( A, s, t, k )
 /* select the order k element in A from subarray A[s,..t-1] */
   if (k < s or k ≥ t or s ≥ t) return None;
   m   = Rand-Partition ( A, s, t );
   pivot_order = m+1 ;
   if ( pivot_order = k)  return A[m];
   if ( pivot_order > k )
        return Rand-Select ( A, s, m, k );
   else  return Rand-Select ( A, m+1, t, k );
```

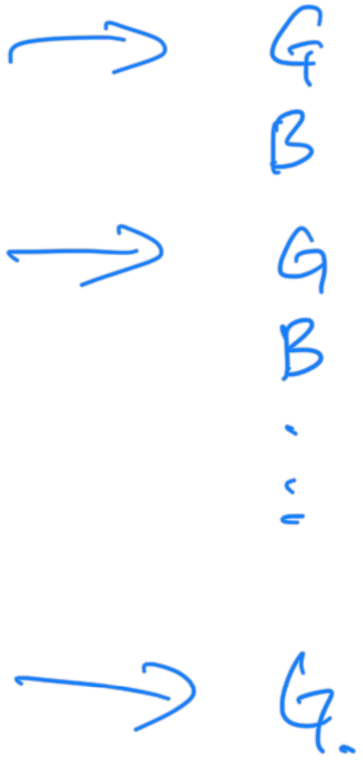
Rand-Partition(A, s, t) uses a random element from A[s, … t-1] as pivot, instead of using A[t-1] as pivot like in Partition(A, s, t).

Rand-Partition pseudo-code

```
Rand-Partition(A, s, t)
/* Partition the subarray A[s, ..., t − 1] using a random pivot.
/* ℓ: index for mid_barrier index; and r: index for end_barrier.
pivot_id = random(s, t);
p = A[pivot_id];
exchange A[pivot_id] with A[t − 1];
ℓ = s;
for r = s to t − 2 do
    if A[r] ≤ p then
        exchange A[ℓ] with A[r];
        ℓ++;
    end
end
exchange A[ℓ] with A[t − 1];
return (ℓ);
```

Expected time analysis -- intuition

- In expectation, after every constant number of recursive calls, there will be a good split,
  - Good split:
    - the pivot has rank in $[\frac{n}{4}, \frac{3n}{4}]$ => probability of a good split $p = \frac{1}{2}$
  - Bad split:
    - Otherwise



- Every time a good split happens,
  - the size of the problem will be reduced by at least $\frac{1}{4}$
  - i.e, the remainder size is at most $\frac{3}{4}n'$ where n' is the previous size

Expected time analysis -- intuition

- Recall $T(n) = \max(T(r-1), T(n-r)) + cn$
- Counting the cost of all good splits, we have that it is at most
  - $T_{good}(n) \leq T_{good}\left(\frac{3n}{4}\right) + cn$
  - $\Rightarrow T_{good}(n) \leq cn + \frac{3}{4}cn + \left(\frac{3}{4}\right)^2 cn + \ldots = cn\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \ldots\right) = \Theta(n)$
- In-between good splits there are bad splits, but their costs intuitively can be charged to those of the good splits
  - The good split happens with probability $p = \frac{1}{2}$
  - Expected cost of bad splits is bounded by $(\frac{1-p}{p})T_{good}(n) = T_{good}(n)$
- Hence the expected total time is $\mathrm{E}T(n) \leq 2T_{good}(n) = \Theta(n)$

Expected time analysis -- intuition

- Recall $T(n) = \max\big(T(r-1), T(n-r)\big) + \ cn$
- Counting the cost of all good splits, we have that it is at most
  - $T_{good}(n) \le T_{good}\left(\frac{3n}{4}\right) + cn$
  - $\Rightarrow T_{good}(n) \le cn + \frac{3}{4}cn + \left(\frac{3}{4}\right)^2 cn + \ \ldots = cn\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \ \ldots\right) = \Theta(n)$
- In-between good splits there are bad splits, but their costs intuitively can be charged to those of the good splits
  - The good split happens with probability $p = \frac{1}{2}$
  - Expected cost of bad splits is bounded by $(\frac{1-p}{p})T_{good}(n) = T_{good}(n)$
- Hence the expected total time is $\mathrm{E}T(n) \le 2T_{good}(n) = \Theta(n)$

This is NOT a precise argument, just intuition.
This can be made more precise.

Summary

- Randomized version of QuickSelect runs in $\Theta(n)$ expected time
- In fact, one can perform Select in $\Theta(n)$ worst-case time
  - Not covered in this class.

Part E:
Randomized QuickSort

Sorting revisited!

- Previously, MergeSort
  - Divide and conquer paradigm
  - But NOT in-place sorting

- Now: QuickSort
  - In-place sorting
  - Randomized quicksort:
    - Worst case: $\Theta(n^2)$
    - Expected running time: $\Theta(n \lg n)$

Recall MergeSort

```
MergeSort ( A, r, s ) // sorting subarray A[r,s]

  if ( r ≥ s)  return;
  m  = (r+s) / 2;
  A1 = MergeSort ( A, r, m );
  A2 = MergeSort ( A, m+1, s );

  Merge (A1,A2);
```

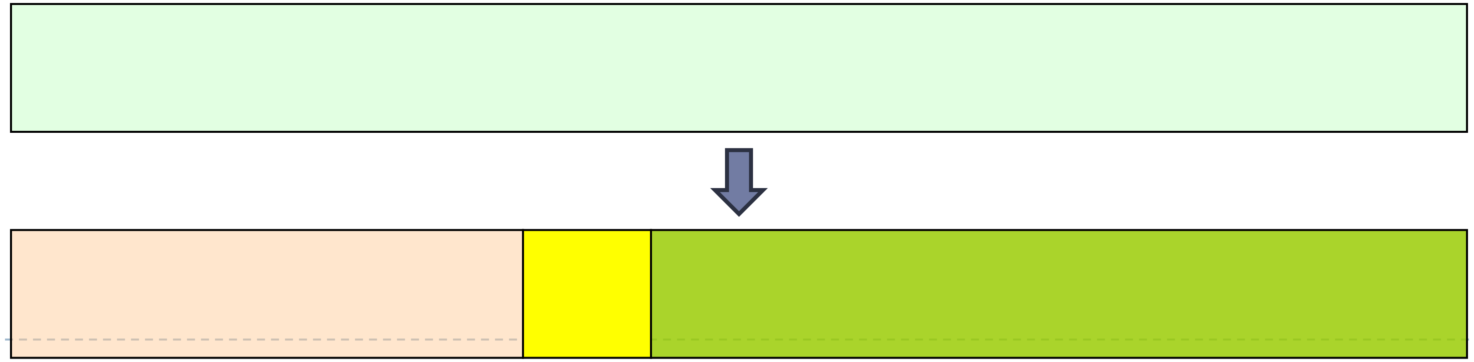
- Much work has to be done in Merge(), but the "divide" step is easy (simply split the array into two equal parts).

QuickSort

```
QuickSort ( A, r, s )

   if ( r ≥ s)  return;
   m   = Partition ( A, r, s );
   A1  = QuickSort ( A, r, m );
   A2  = QuickSort ( A, m+1, s );

   Merge (A1,A2);
```

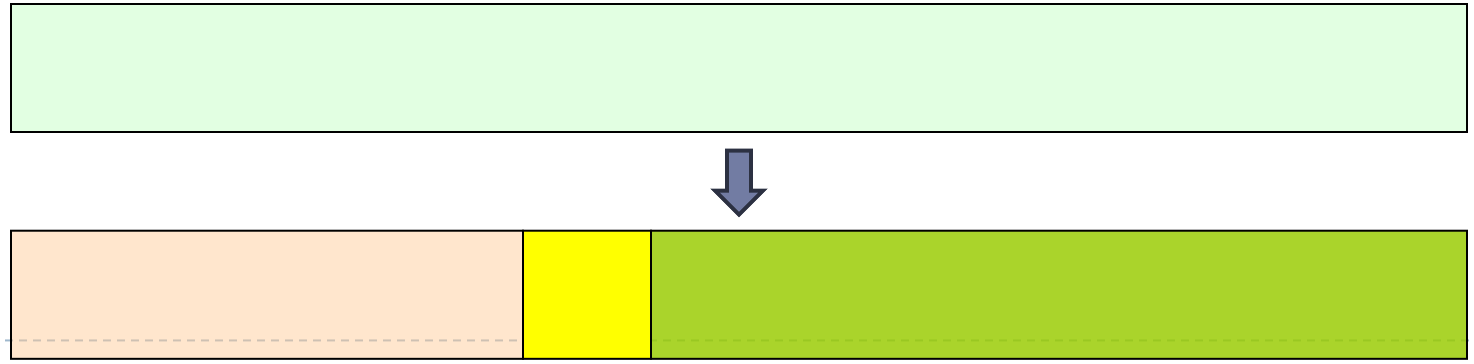


A[m]: pivot

QuickSort

```
QuickSort ( A, r, s )

  if ( r ≥ s)  return;
  m   = Partition ( A, r, s );
  A1 = QuickSort ( A, r, m );
  A2 = QuickSort ( A, m+1, s );
  Merge (A1,A2);
```



A[m]: pivot

QuickSort

```
QuickSort ( A, r, s )

   if ( r ≥ s)  return;
   m   = Partition ( A, r, s );
   A1  = QuickSort ( A, r, m-1 );
   A2  = QuickSort ( A, m+1, s );
```

- Worst case

- Best case
  -

QuickSort

```
QuickSort ( A, r, s )

  if ( r ≥ s)  return;
  m  = Partition ( A, r, s );
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```

- Worst case
  - $T(n) = T(n-1) + cn = \Theta(n^2)$
- Best case
  - $T(n) = 2T\left(\frac{n}{2}\right) + cn = \Theta(n \lg n)$

rand-QuickSort

```
rand-QuickSort ( A, r, s )

   if ( r ≥ s)  return;
   m   = rand-Partition ( A, r, s );
   A1 = rand-QuickSort ( A, r, m-1 );
   A2 = rand-QuickSort ( A, m+1, s );
```

rand-QuickSort

```
rand-QuickSort ( A, r, s )

   if ( r ≥ s)  return;
   m   = rand-Partition ( A, r, s );
   A1  = rand-QuickSort ( A, r, m-1 );
   A2  = rand-QuickSort ( A, m+1, s );
```

- Worst case
  - $T(n) = T(n-1) + cn = \Theta(n^2)$
- Best case
  - $T(n) = 2T\left(\frac{n}{2}\right) + cn = \Theta(n \lg n)$

- rand-QuickSelect
  - like rand-Select, there are good and bad splits
  - as long as good splits come constant fraction of the time, the time complexity is dominated by good splits
  - expected running time is $ET(n) = \Theta(n \lg n)$

- rand-QuickSelect
  - like rand-Select, there are good and bad splits
  - as long as good splits come constant fraction of the time, the time complexity is dominated by good splits
  - expected running time is $ET(n) = \Theta(n \lg n)$

- Compared to MergeSort
  - In-place sorting
    - while MergeSort needs to open a new output array of size $\Theta(n)$
  - In practice often faster, and needs much smaller memory (important!)

FIN