DSC40B:
Theoretical Foundations of Data Science II

Lecture 8: *Binary search tree*

Instructor: Yusu Wang

(Dynamic) Set operations

- Imagine you are maintaining a database indexed by some keys (real values), and you hope to support the following operations:

  - Search
  - Maximum
  - Minimum
  - Successor
  - Predecessor

  - Insert
  - Delete
  - Extract-Max
  - Increase-key

(Dynamic) Set operations

- Imagine you are maintaining a database indexed by some keys (real values), and you hope to support the following operations:

  - Search
  - Maximum
  - Minimum
  - Successor
  - Predecessor

Static operations!

  - Insert
  - Delete
  - Extract-Max
  - Increase-key

Dynamic operations!

A[k]

(Dynamic) Set operations

- Imagine you are maintaining a database indexed by some keys (real values), and you hope to support the following operations:

Operation	First approach: sort the array of keys
Search	$\Theta(\lg n)$
Maximum	$\Theta(1)$
Minimum	$\Theta(1)$
Successor	$\Theta(1)$
Predecessor	$\Theta(1)$
Insert	
Delete	$\Theta(n)$
Extract-Max	$\Theta(n)$
Increase-key	

(Dynamic) Set operations

- Imagine you are maintaining a database indexed by some keys (real values), and you hope to support the following operations:

First approach: sort the array of keys

Operation	Sorted array
Search	$\Theta(\lg n)$
Maximum	$\Theta(1)$
Minimum	$\Theta(1)$
Successor	$\Theta(1)$
Predecessor	$\Theta(1)$
Insert	
Delete	$\Theta(n)$
Extract-Max	$\Theta(n)$
Increase-key	

Using a sorted array can handle all static operations efficiently

How to have a good data structure so we can support all these operations efficiently?

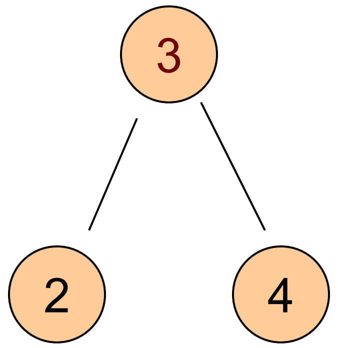
Today

- Binary search tree
  - support all the operations from previous slide
    - in time proportional to height of tree
- (Review): how to implement key operations, and time complexity
  - search, insert (and delete)
- Extension to balanced binary search tree
- *Select* query: augmenting data structure
  - median, order statistics

Part A:
What is binary search tree?

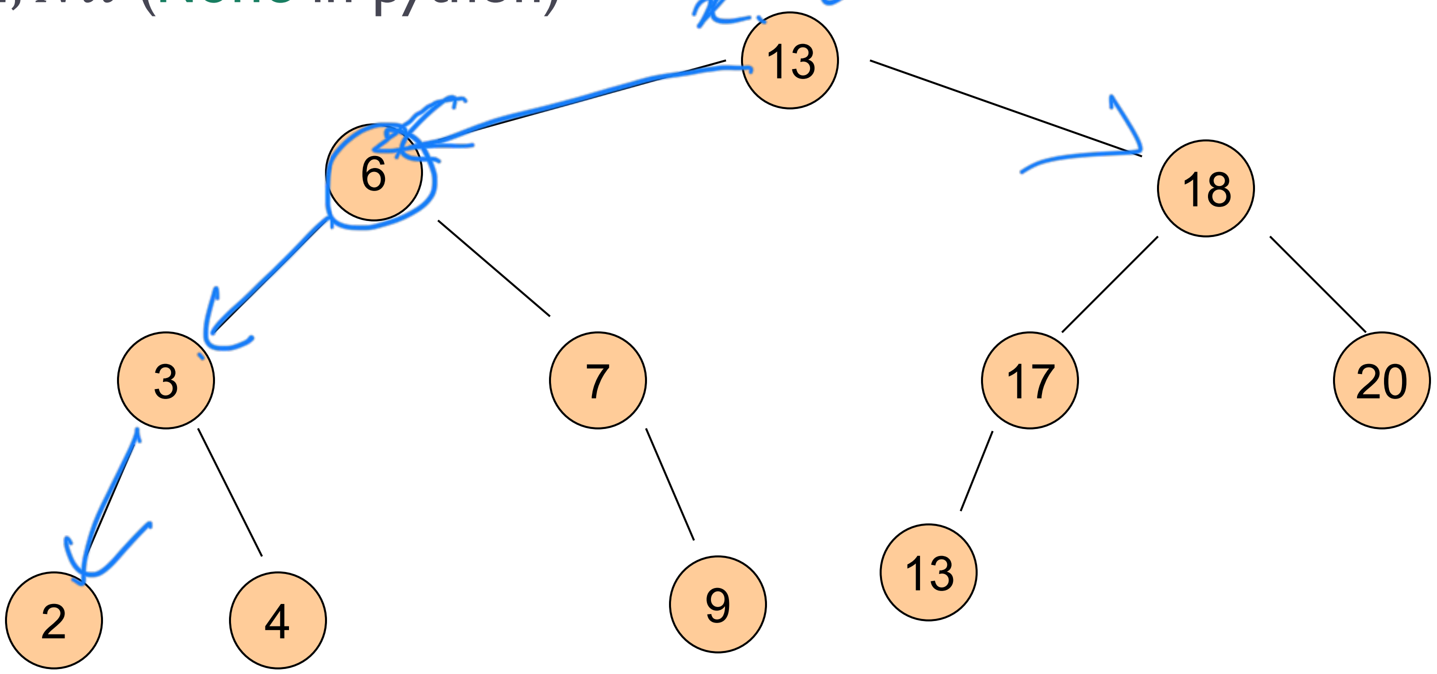
First: Binary tree

- A binary tree is a rooted tree
  - where each node has at most 2 children
- Represented by a linked data structure
- Each node contains at least fields:
  - *Key*
  - *Left*
  - *Right*
  - *Parent*



Example

- From root, following left pointers, we will visit
  - $13, 6, 3, 2, Nil$ (None in python)



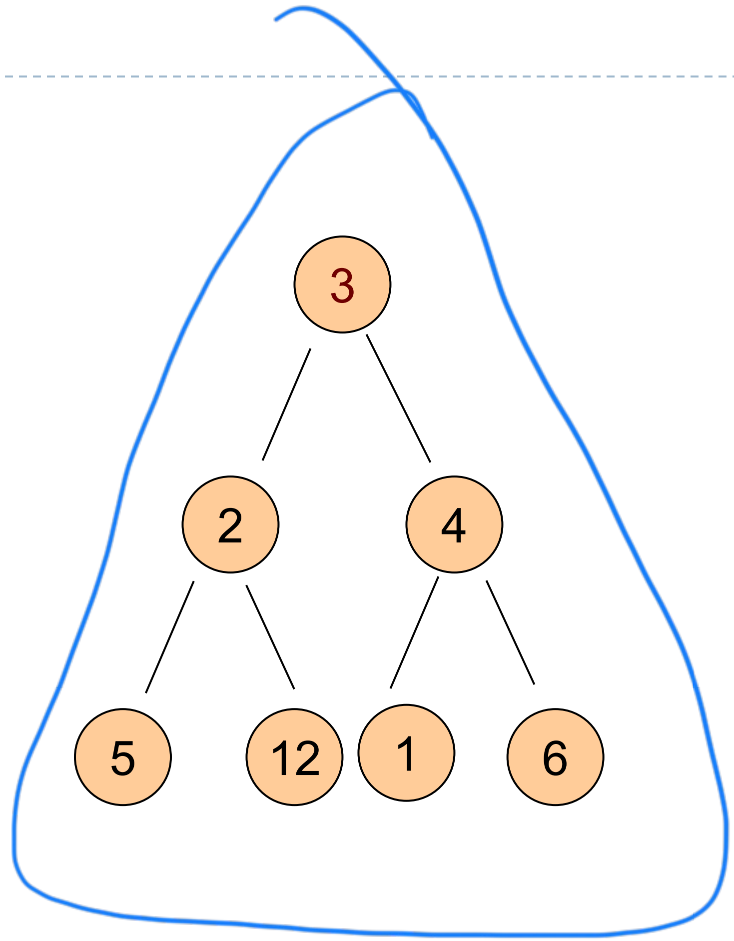
Create a single node tree in Python

```
class Node:
    def __init__(self, key, parent=None):
        self.key = key
        self.parent = parent
        self.left = None
        self.right = None

class BinarySearchTree:
    def __init__(self, root: Node):
        self.root = root
```

Binary tree

- A binary tree is a rooted tree where
  - each node has at most 2 children
- A node is the root of the tree
  - if its parent is Nil
- A node is a leaf
  - if both children are Nil
- Left sub-tree, right sub-tree

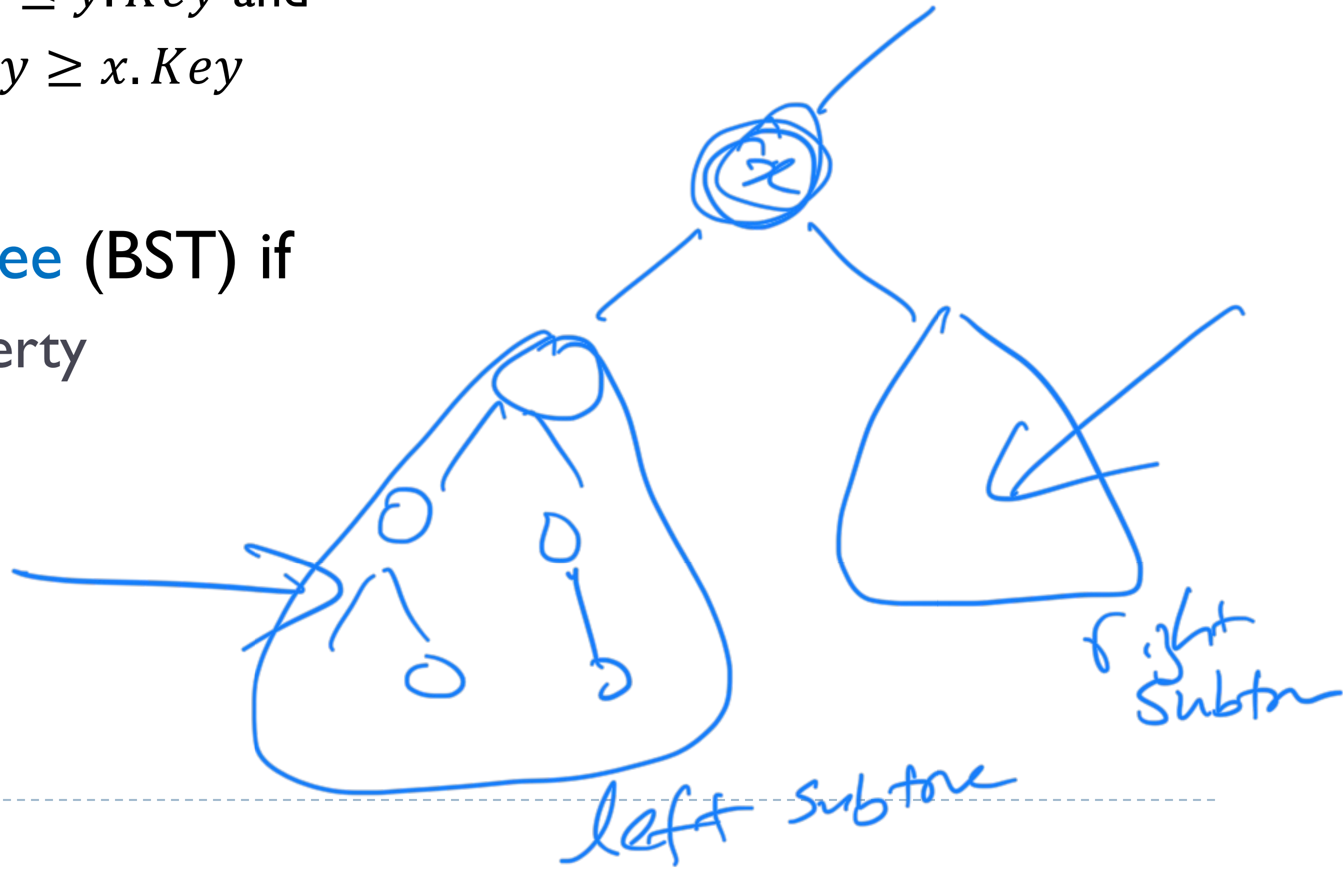


- A complete binary tree is a binary tree
  - where each node has two children other than leaves
  - and each level (except possibly last level) is filled, and all nodes are as left as possible.

Binary search tree (BST)

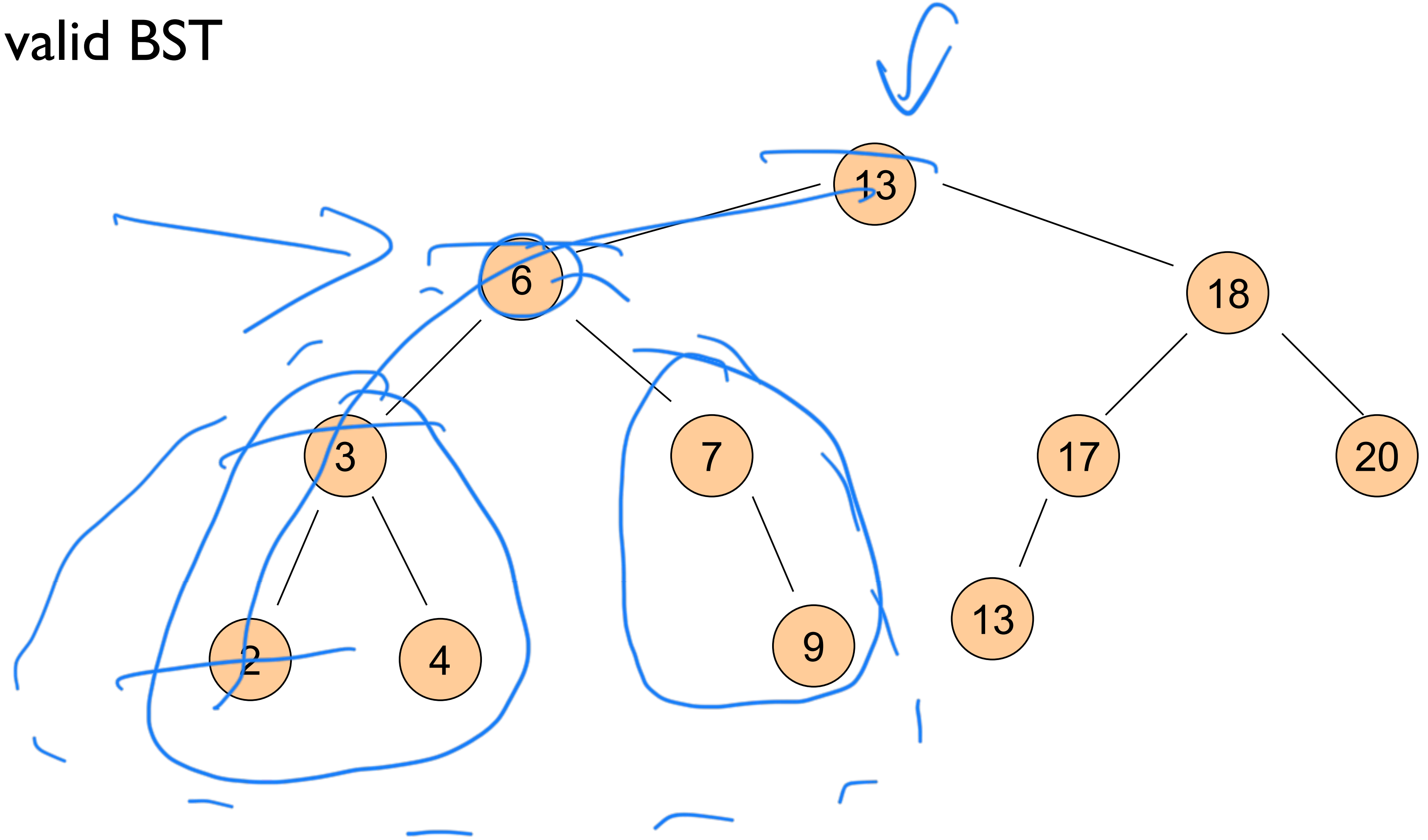
- Binary-search-tree property
  - For any node $x \in T$,
    - if y is in the left subtree of x, then $y.Key \le y.Key$ and
    - if y is in the right subtree of x, then $y.Key \ge x.Key$

- A binary tree T is a binary search tree (BST) if
  - it satisfies the binary search tree property



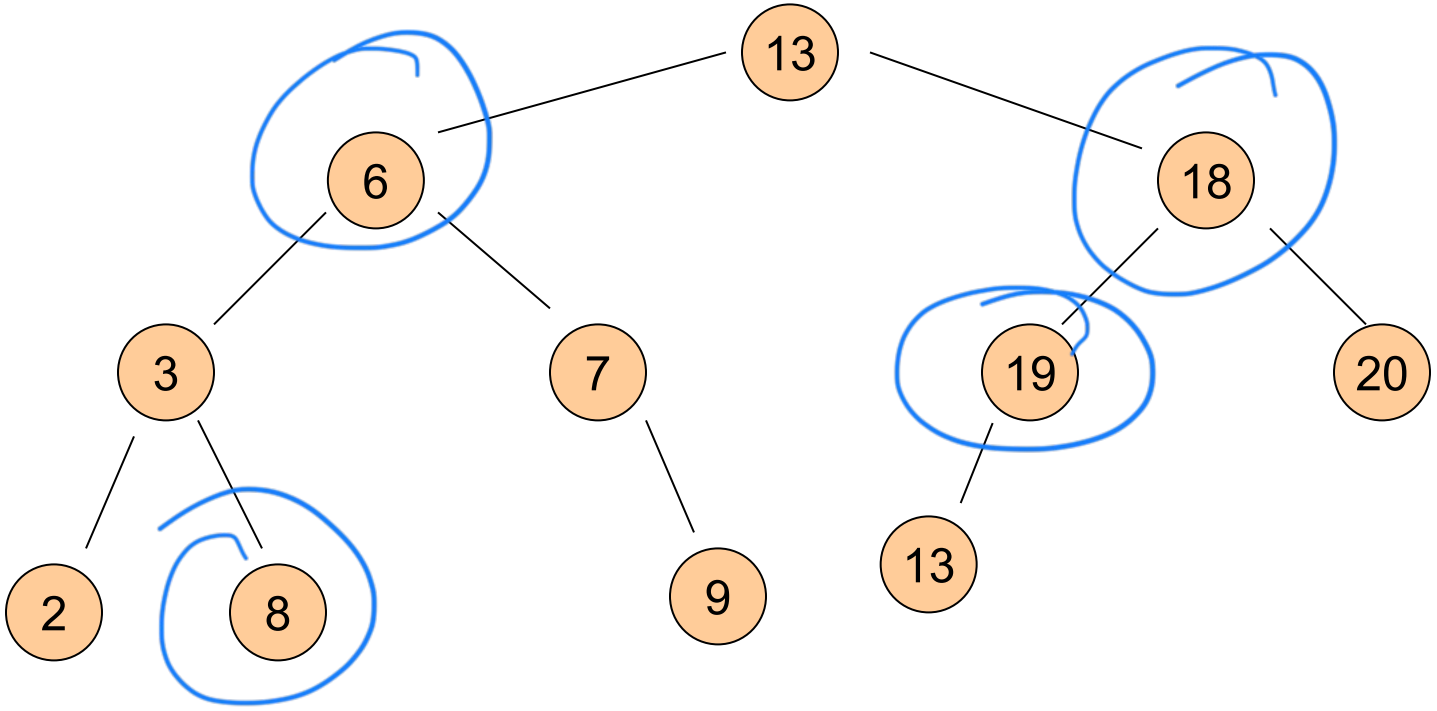
Example

- A valid BST



Example

- ?



Properties

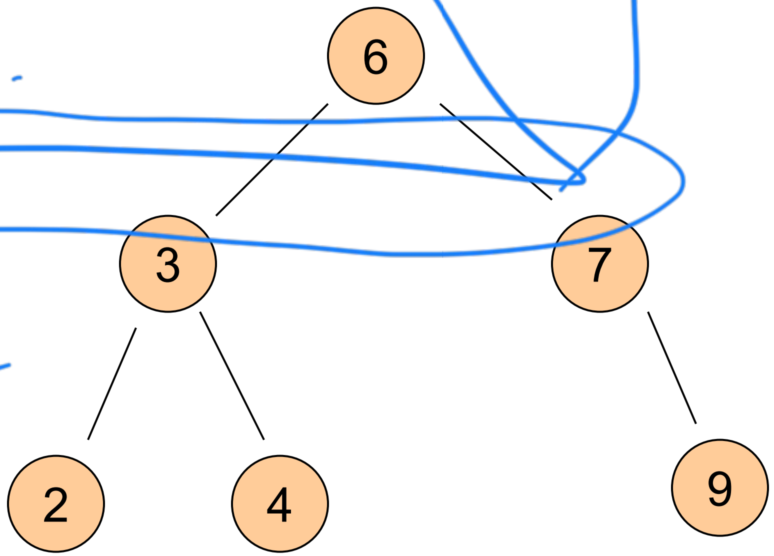
- Given the same set of elements
  - there are many possible BSTs over them
- Given n nodes,
  - Tallest possible BST tree has height $h =$ n
  - Shortest possible BST tree has height $h =$ $\log_2 n$

3, 10, 5, 4, 9

$1 + 2 + 2^2 + \cdots + 2^k = n$

$2^{k+1} - 1 = n$

$\Rightarrow k + 1 = \log_2 n$



Properties

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- Given n nodes,
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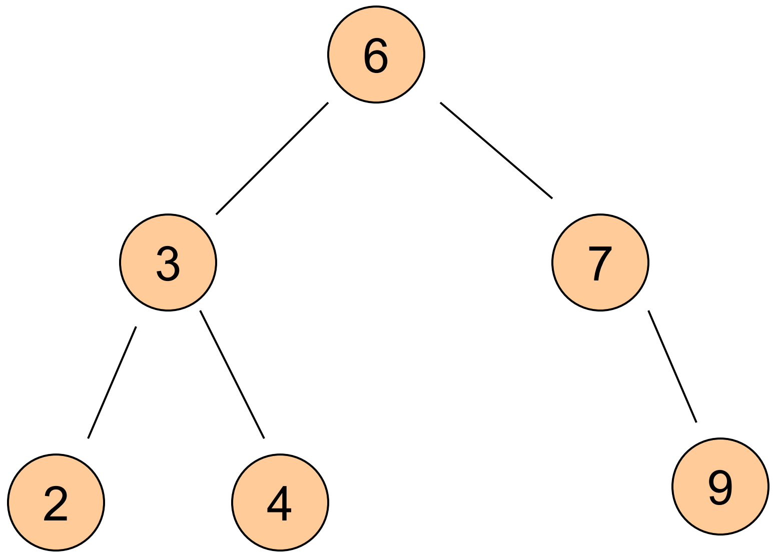
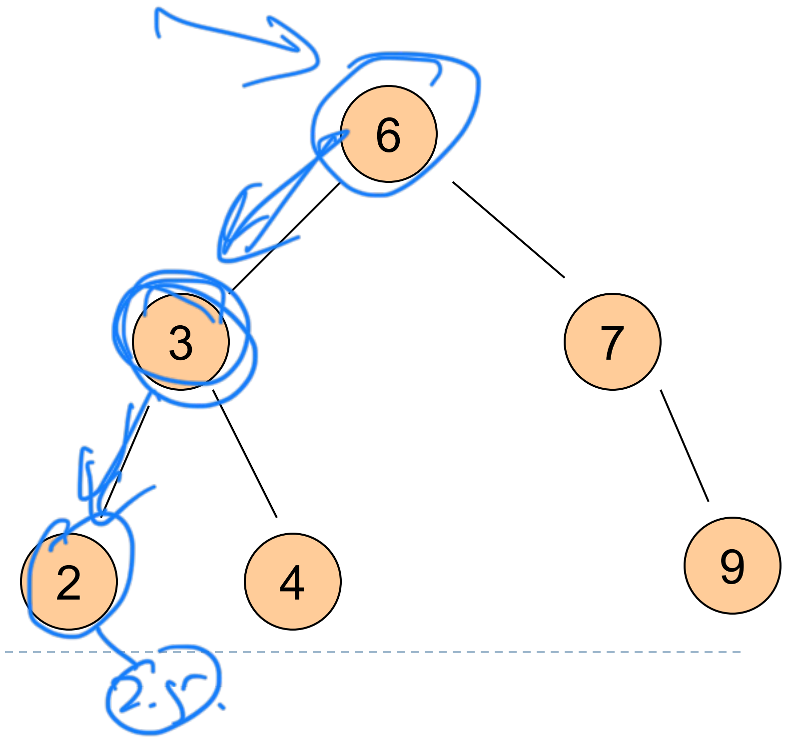


Properties

- Given the same set of elements
  - there are many possible BSTs over them
- Given n nodes,
  - Tallest possible BST tree has height $h =$ n
  - Shortest possible BST tree has height $h =$ $\log_2 n = \Theta(\lg n)$
- Minimum?
  - Does it have to be a leaf?
- Maximum?

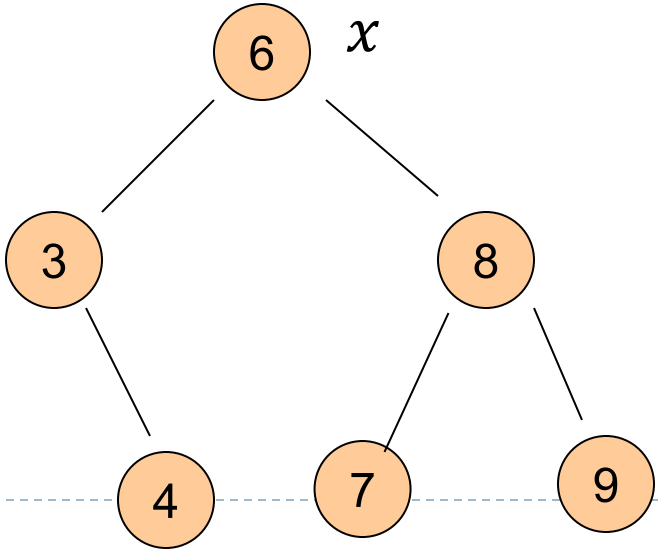


Part B:
Operations in BST

Search operation

- A BST T with n nodes can be viewed as a way to store n keys in a smart way, so that queries among these keys become easy.

- Tree-search(x, k)
  - Input: given a tree node x and a query key k
  - Output: search whether k is in the tree rooted at x
    - if it is in, return a node y s.t. $y.key = k$
    - otherwise, returns NIL



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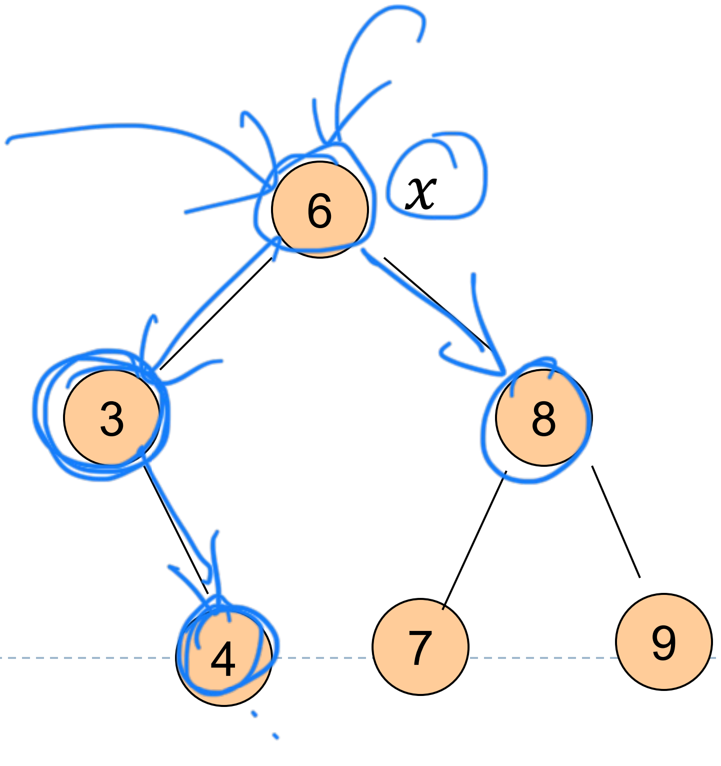
Tree-search$(x, 8)$

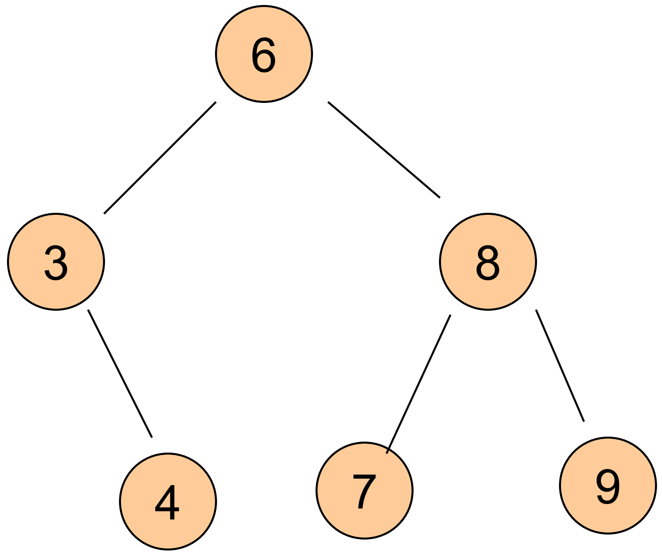
Tree-search$(x, 4)$

Tree-search$(x, 5)$ Nil



Tree-search algorithm, recursive version

```
Tree-search ( x, k)
  if x is None:
    return False
  if x.key == k
    return x
  elif k < x.key
    return Tree-search( x.left, k )
  else:
    return Tree-search( x.right, k )
```



- Given an input tree T and a key k
  - we will start by calling Tree-search(T.root, k)

Tree-search algorithm, recursive version

```
Tree-search ( x, k)
  if x is None:
     return False
  if x.key == k
     return x
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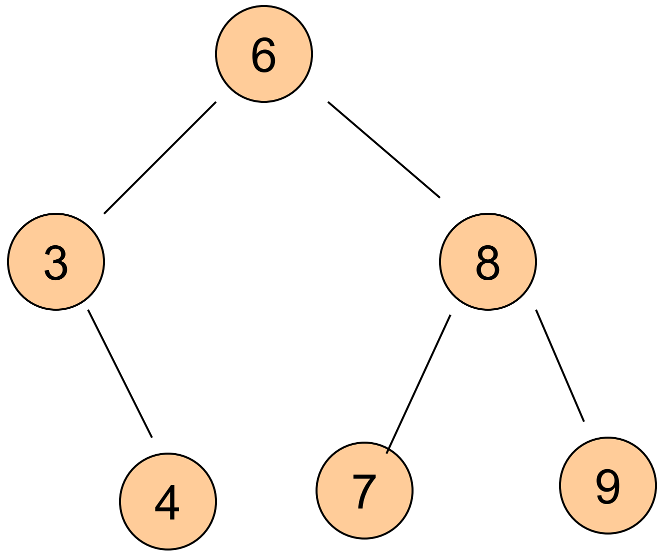

- Time complexity analysis
  - let $T(n)$ denote the worst case time complexity of procedure Tree-search() on any tree of n nodes

Tree-search algorithm, recursive version

```
Tree-search ( x, k)
  if x is None:
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  if x.key == k
     return x
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       return Tree-search( x.left, k )
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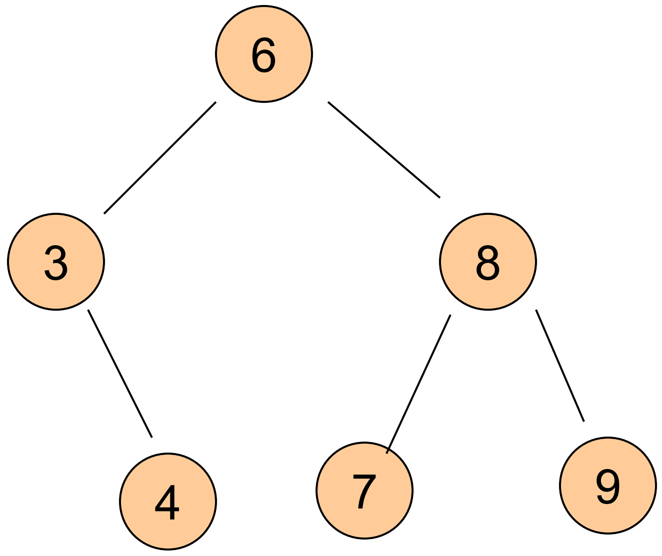


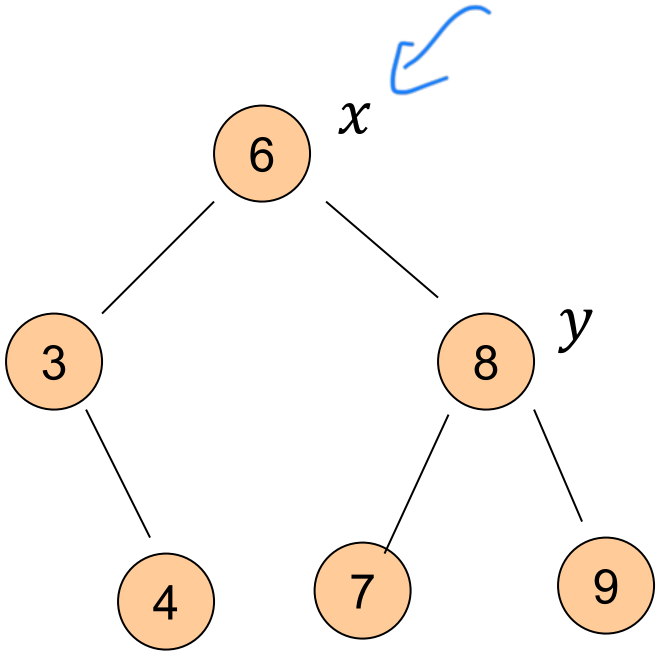
- Time complexity analysis
  - Other than recursive call, $\Theta(1)$ within each Tree-search call. Hence $T(n)$ is proportional to the number of nodes x we will call Tree-search on. This implies $T(n) = \Theta(\text{tree-height}) = O(n)$

Tree-search: Pseudo-code of the iterative version

```
procedure IterativeTreeSearch(x,K)
while (x = NIL) and (K ≠ x.key) do
    if (K ≤ x.key) then
        x ← x.left;
    else
        x ← x.right;
    end
end
return (x);
```

Minimum / Maximum

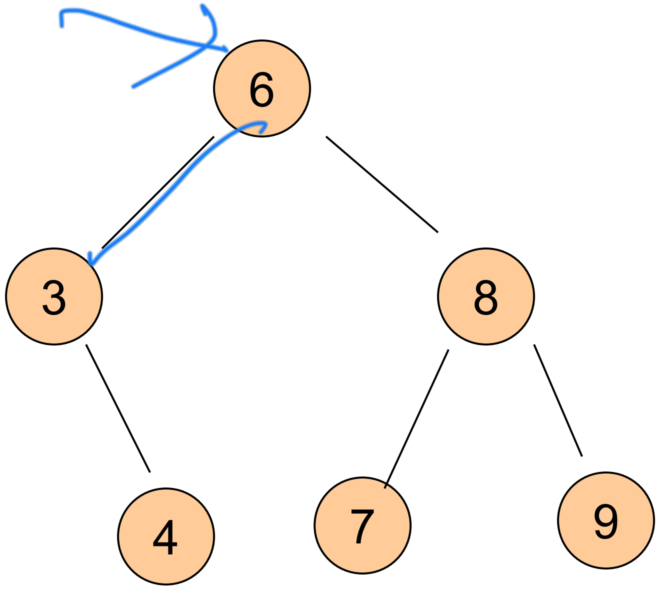
- Tree-minimum(x)
  - Input: a node x of a BST T
  - Output: return the node containing minimum key in the subtree rooted at x



Minimum / Maximum

- Tree-minimum(x)
  - Input: a node x of a BST T
  - Output: return the node containing a minimum key in the subtree rooted at x

```
Tree-minimum(x)
    while x.left is not None
        x = x.left;
    return x;
```



Minimum / Maximum

- Tree-minimum(x)
  - Input: a node x of a BST T
  - Output: return the node containing a minimum key in the subtree rooted at x

```
Tree-minimum(x)
        while x.left is not None
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        return x;
```
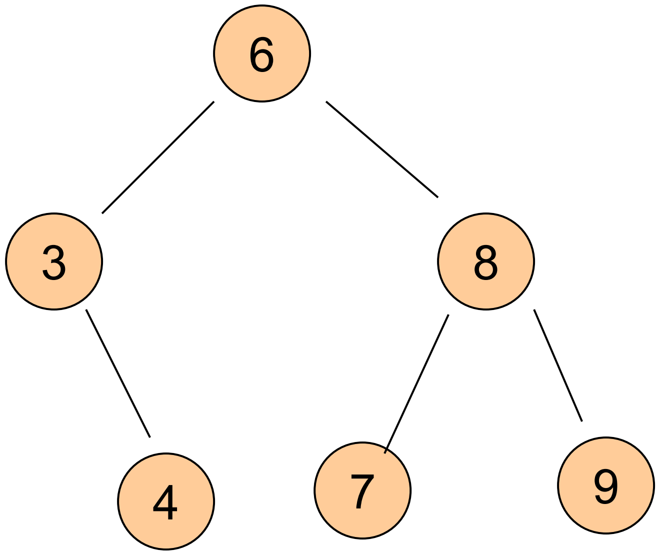
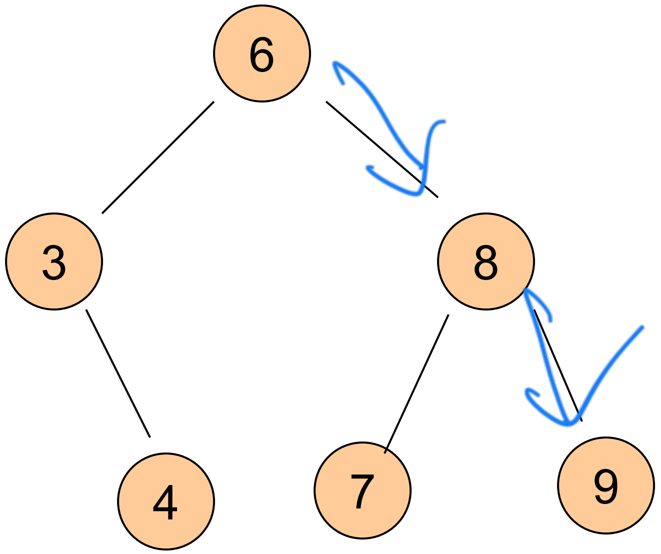



- Time complexity
  - $T(n) = \Theta(h)$ where h is height of input tree

Minimum / Maximum

- Tree-maximum(x)
  - Input: a node x of a BST T
  - Output: return the node containing a maximum key in the subtree rooted at x

```
Tree-maximum(x)
    while x.right is not None
        x = x.right;
    return x;
```

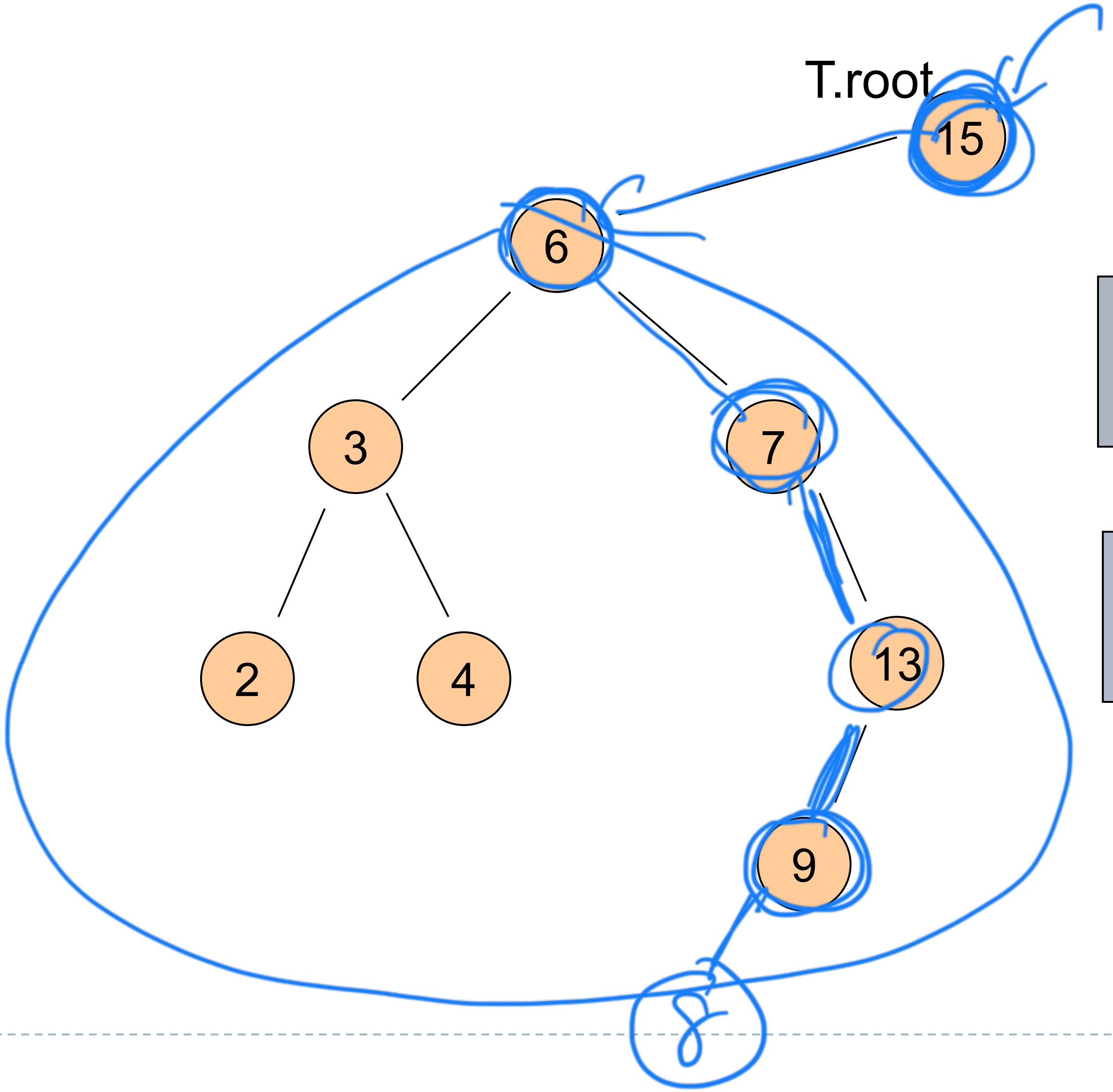


- Time complexity
  - $T(n) = \Theta(h)$ where h is height of input tree

Tree-insert

- Tree-insert(x, k)
  - Input: a BST tree node x and a key k
  - Output: insert k to the tree rooted at x such that the resulting tree is still a binary search tree

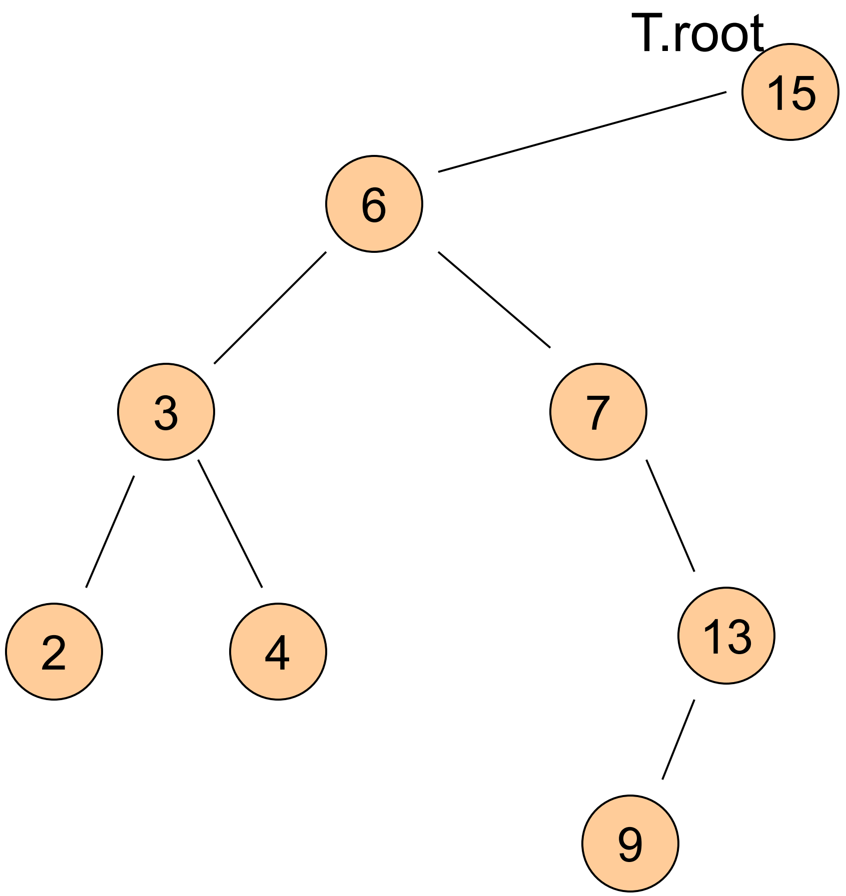
Examples



Tree-Insert(T.root, 8)

Tree-Insert(T.root, 6.5)

Examples



Tree-Insert(T.root, 8)

Tree-Insert(T.root, 6.5)

Use tree-search !

Pseudo-code of Tree-insert

```
Tree-insert(T, k)
    y = Nil;   x = T.root
    z.key = k; z.left = Nil; z.right=Nil
    while (x ≠ Nil)  do
            y = x
            if  ( z.key < x.key )
                then  x = x.left
                else  x = x.right
    z.parent = y
    if  (y = Nil)  then  T.root = z
    else  if   (z.key < y.key)
            then      y.left = z
            else      y.right = z
```

Tree-insert

Tree-insert(T, k)

```
y = Nil;   x = T.root
z.key = k; z.left = Nil; z.right=Nil
while (x ≠ Nil)  do
        y = x
        if ( z.key < x.key )
            then  x = x.left
            else  x = x.right
z.parent = y
if (y = Nil)  then T.root = z
else if  (z.key < y.key)
        then    y.left = z
        else    y.right = z
```

- z is the new node to be inserted
- Locate potential parent y of z.

- Set up z as appropriate child of y

Tree-insert

```
Tree-insert(T, k)
   y = Nil;   x = T.root
   z.key = k; z.left = Nil; z.right=Nil
   while (x ≠ Nil) do
        y = x
        if ( z.key < x.key )
            then  x = x.left
            else  x = x.right
   z.parent = y
   if (y = Nil) then T.root = z
   else if (z.key < y.key)
        then   y.left = z
        else   y.right = z
```

- Time complexity
  - $T(n) = \Theta(h)$, where h is height of input tree

Tree Insert Python Code

```python
def insert(self, new_key):
    # assume new_key is unique
    current_node = self.root
    parent = None

    # find place to insert the new node
    while current_node is not None:
        parent = current_node
        if current_node.key < new_key:
            current_node = current_node.right
        else: # current_node.key > new_key
            current_node = current_node.left

    # create the new node
    new_node = Node(key=new_key, parent=parent)

    # if parent is None, this is the root. Otherwise, update the
    # parent's left or right child as appropriate
    if parent is None:
        self.root = new_node
    elif parent.key < new_key:
        parent.right = new_node
    else:
        parent.left = new_node
```

Summary: BST is good for both static and dynamic operations

- Suppose n input keys are already stored in a BST of height h

Operation	Time complexity
Search	$\Theta(h)$
Maximum	$\Theta(h)$
Minimum	$\Theta(h)$
Successor	$\Theta(h)$
Predecessor	$\Theta(h)$
Insert	$\Theta(h)$
Delete	$\Theta(h)$
Extract-Max	$\Theta(h)$
Increase-key	$\Theta(h)$

- However, performance depending on height!
- Height $h = O(n)$ and $h = \Omega(\lg n)$

- To have good performance, we want to keep the tree height low!

Summary: BST is good for both static and dynamic operations

- Suppose n input keys are already stored in a BST of height h

Operation	Time complexity
Search	$\Theta(h)$
Maximum	$\Theta(h)$
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Successor	$\Theta(h)$
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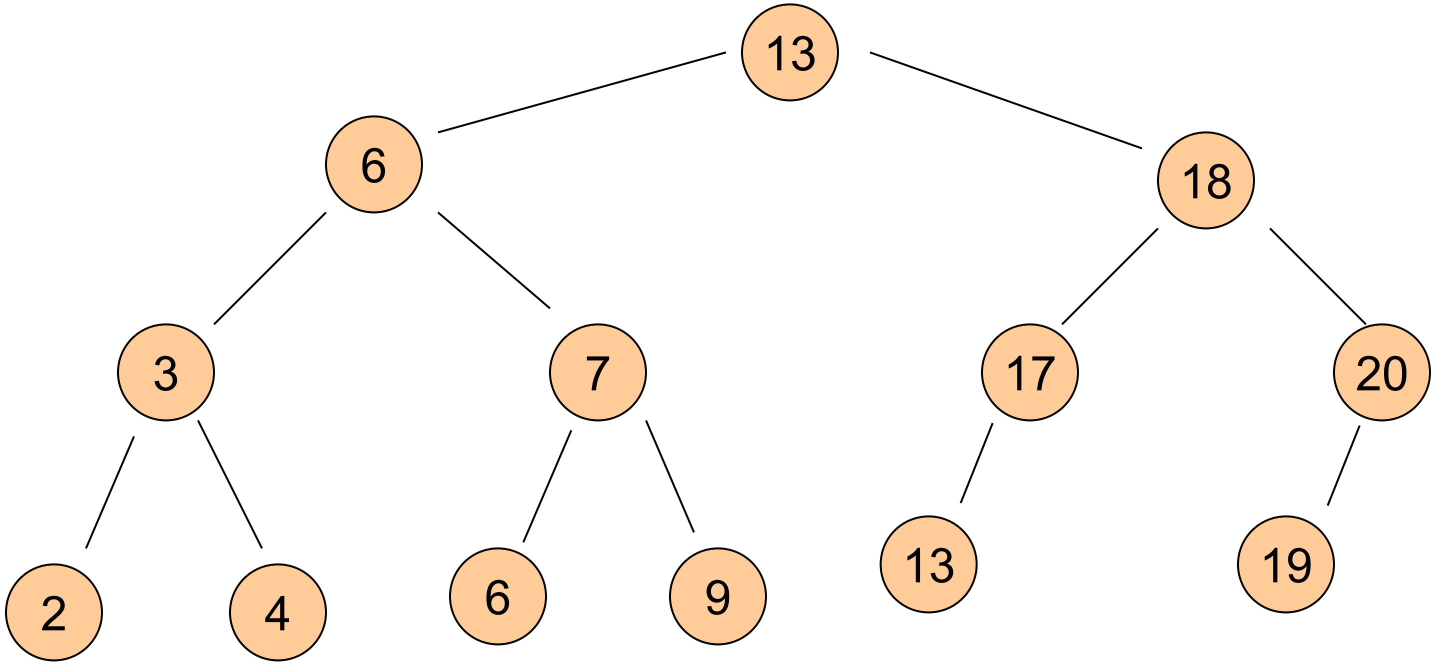
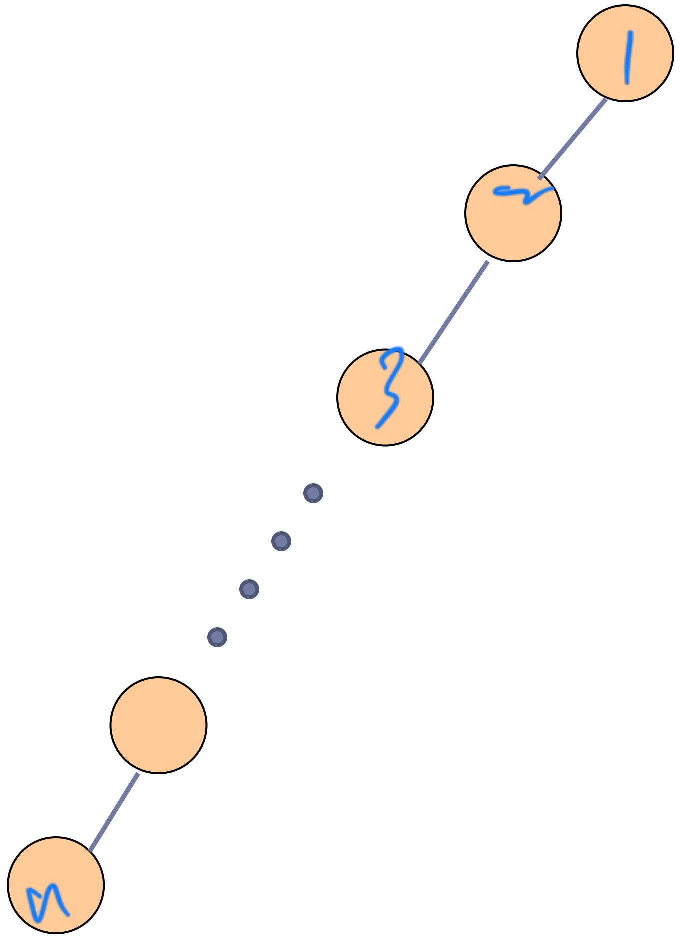
Extract-Max	$\Theta(h)$
Increase-key	$\Theta(h)$

Part C:
Balanced binary search tree

Good tree



Bad Tree



Balanced binary search tree

- It turns out that there are ways to add extra conditions to binary search trees, so that their height is $\Theta(\lg n)$
  - E.g, red-black tree, AVL tree, etc

- Once such a tree is created,
  - it can support search, minimum, maximum etc in $\Theta(h) = \Theta(\lg n)$ time using the same algorithms described before
  - the extra work comes at handling dynamic operations: insertion, deletion, and so on. Re-balancing is needed
  - however, for standard balanced BSTs, all these operations can be handled in $\Theta(\lg n)$ time.

With balanced BST

- Suppose n input keys are already stored in a balanced BST

Operation	Time complexity
Search	$\Theta(\lg n)$
Maximum	$\Theta(\lg n)$
Minimum	$\Theta(\lg n)$
Successor	$\Theta(\lg n)$
Predecessor	$\Theta(\lg n)$
Insert	$\Theta(\lg n)$
Delete	$\Theta(\lg n)$
Extract-Max	$\Theta(\lg n)$
Increase-key	$\Theta(\lg n)$

- Height of tree will be $\Theta(\lg n)$, where n is number of nodes in the tree

Part D:
Select queries
augmenting data structure

- What if we also want to perform Select operation

- BST-Select (*x*, *k*):
  - Given a list of records whose keys are stored in a tree rooted at *x*, return the node whose key has rank *k*.

- We can use QuickSelect to do this in linear time.. Why are we not satisfied?

- What if we also want to perform Select operation

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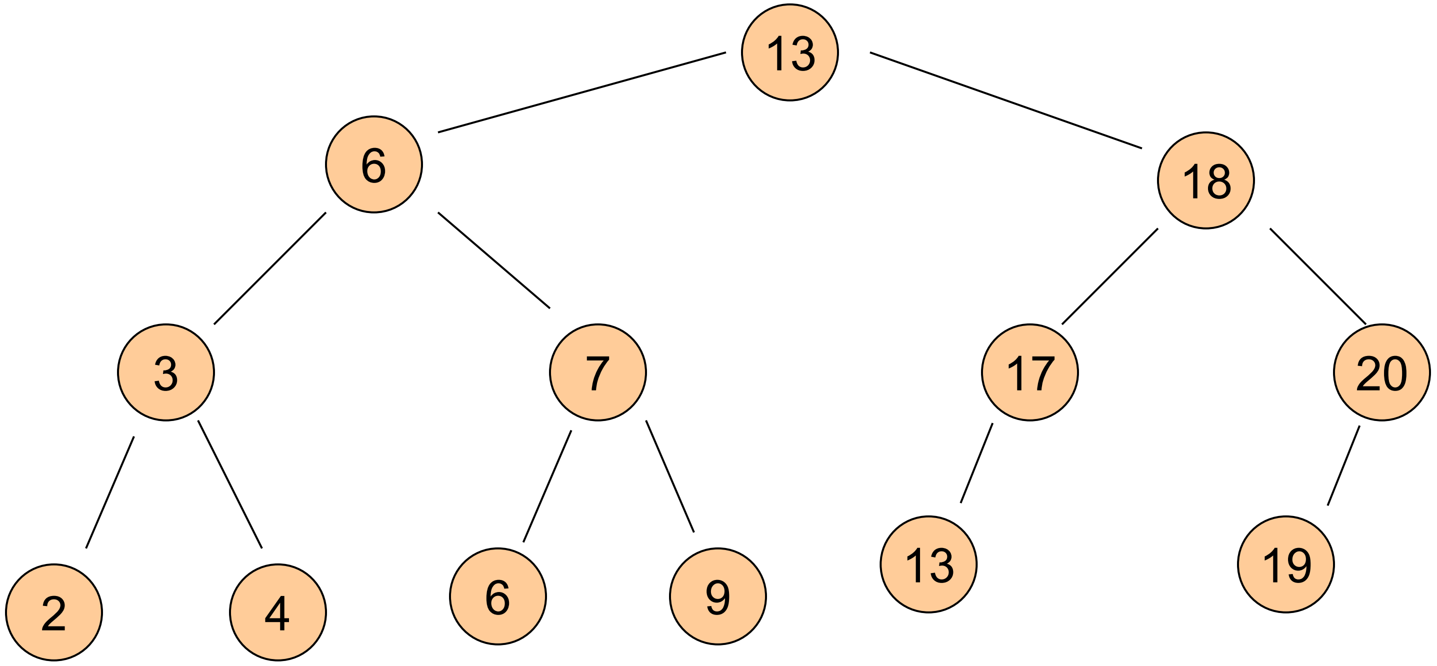
- What if we have to do it many times?
  - Sort it first

- What if we also want to perform Select operation

- BST-Select (*x*, *k*):
  - Given a list of records whose keys are stored in a tree rooted at *x*, return the node whose key has rank *k*.

- We can use QuickSelect to do this in linear time.. Why are we not satisfied?
- What if we have to do it many times?
  - Sort it first
- But what if we also have dynamic changes?
  - Need a data structure that can support Select under dynamic changes

BST



- How to perform Select operation over BST?

- BST-Select (*x*, *k*):
  - Given a list of records whose keys are stored in a tree rooted at *x*, return the node whose key has rank *k*.

- We can do linear search to find it. But can we do better?

- Goal:
  - Augment the binary search tree data structure so as to support Select (*x*, *k*) efficiently

In particular,

- BST-Select (*x*, *k*)
- Goal:
  - Augment the binary search tree data structure so as to support BST-Select (*x*, *k*) efficiently

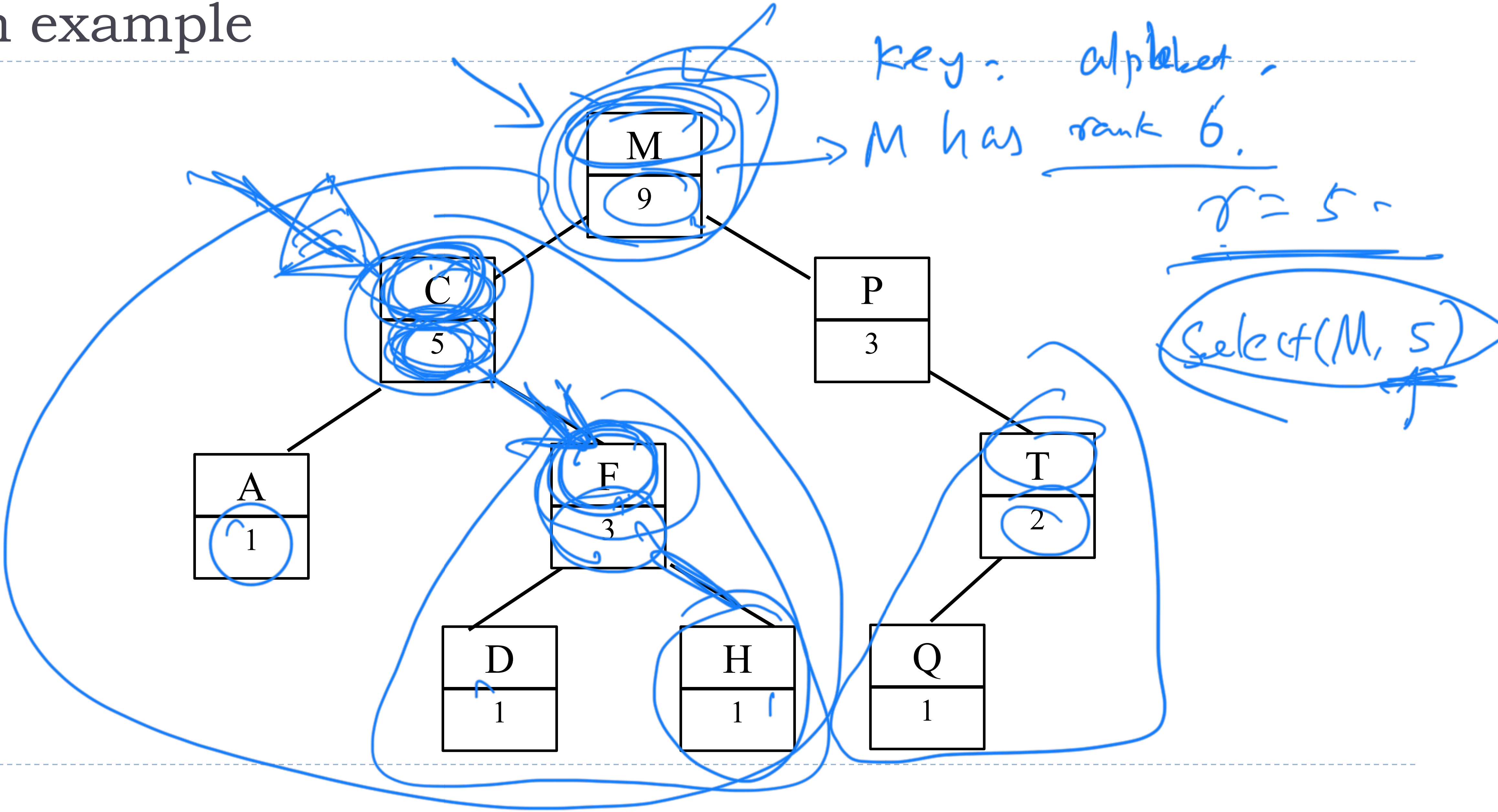
- Ordinary binary search tree T
  - *O(h)* time for BST-Select(*T.root*, *k*) where *h is height of tree T*
- Using balanced search tree)
  - *O(lg n)* time for BST-Select(*T.root*, *k*)

How do we augment a BST T?

- At each node *x* of the tree *T*
  - store *x.size* = # nodes in the subtree rooted at *x*
    - Include x itself
    - If a node (leaf) is NIL, its size is 0.

- Space of an augmented tree:
  - $\Theta(n)$

An example



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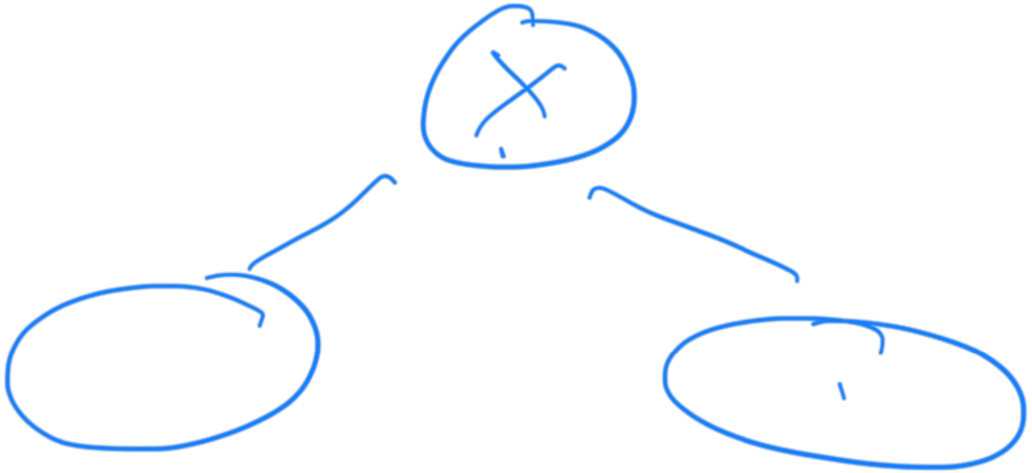
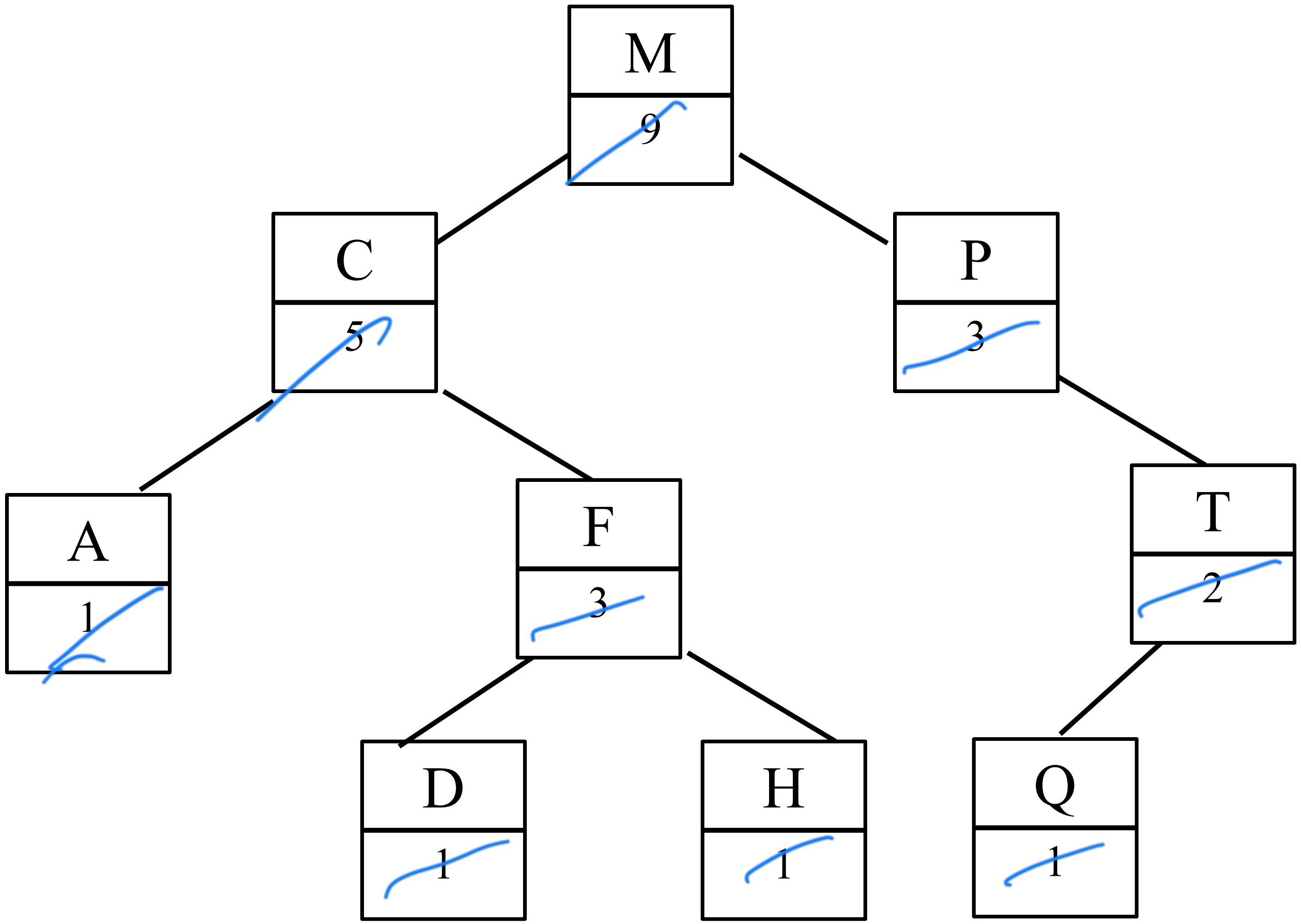
- At each node *x* of the tree *T*
  - store *x.size* = # nodes in the subtree rooted at *x*
    - Include x itself
    - If a node (leaf) is NIL, its size is 0.
- Space of an augmented tree:
  - $\Theta(n)$
- Basic property:
  - $x.size = x.left.size + x.right.size + 1$



How to set up size information ?



How to setup size information?

- procedure AugmentSize(*treenode* x)

  If ($x \neq NIL$) then

  $Lsize$ = AugmentSize($x.left$);

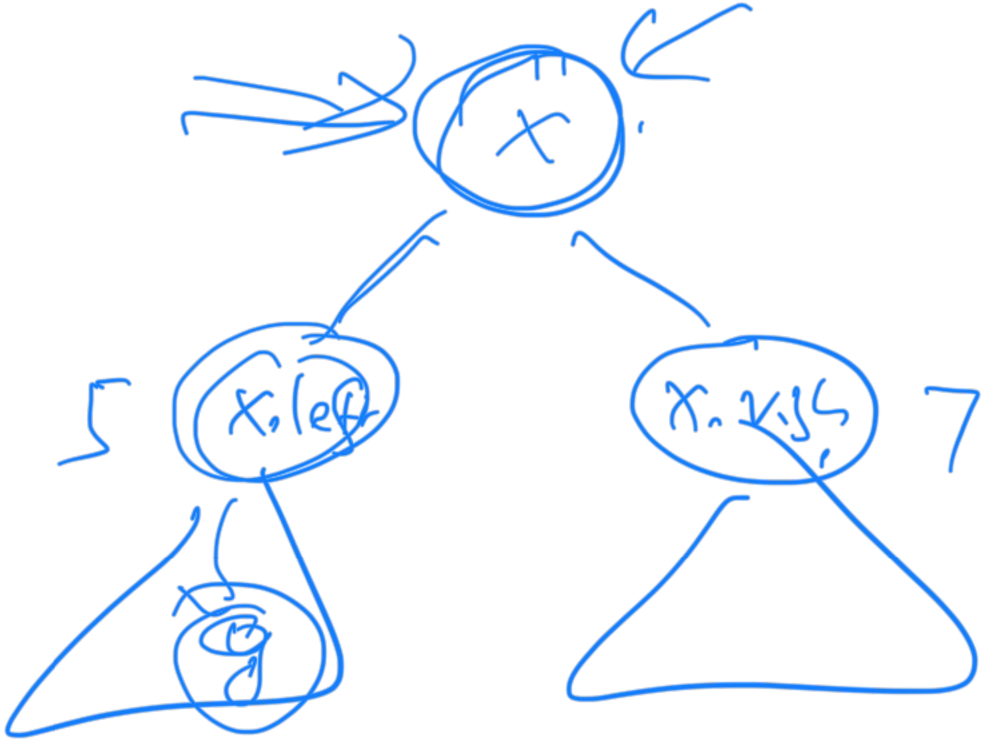
  $Rsize$ = AugmentSize($x.right$);

  $x.size = Lsize + Rsize + 1$;

  Return($x.size$);

  end

  Return (0);



How to setup size information?

- procedure *AugmentSize*($treenode\ \ x$)

  If (x is not *Nil*) then

  $Lsize$ = *AugmentSize*($x.left$);

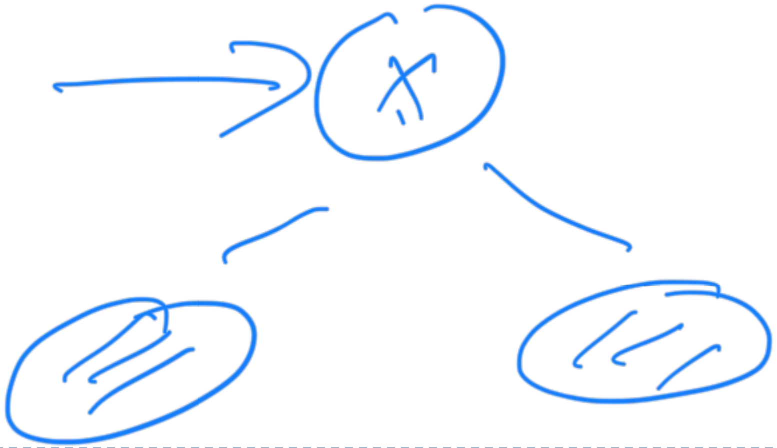
  $Rsize$ = *AugmentSize*($x.right$);

  $x.size = Lsize + Rsize + 1;$

  Return($x.size$);

  end

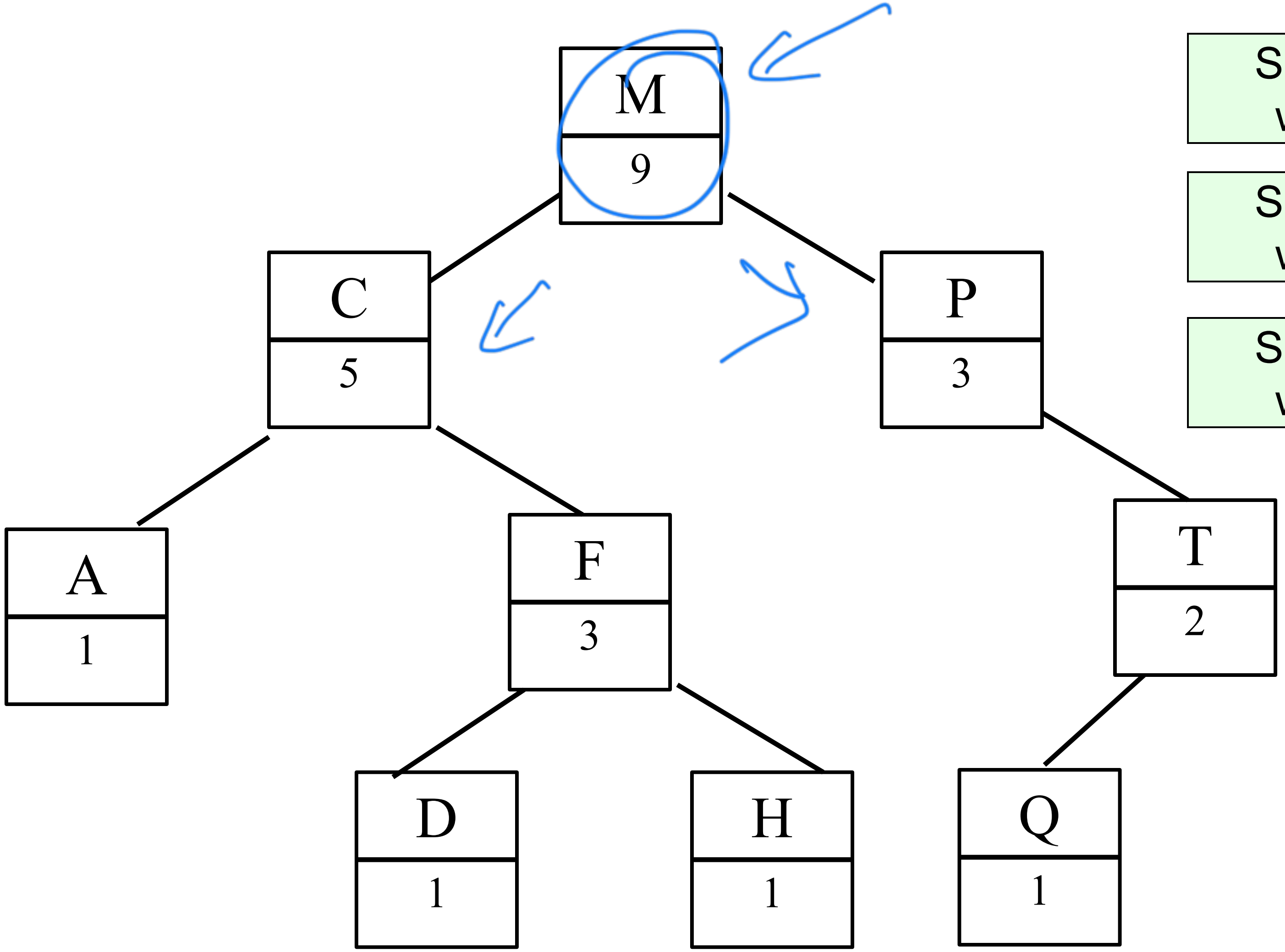
  Return (0);



Postorder traversal
of the tree !

Time complexity for Augmentsize:
$\Theta(size\ of\ tree)$

How to perform select with aug-BST?



Select element with rank 3?

Select element with rank 7?

Select element with rank 6?

- Let *T* be an augmented binary search tree
- *BST-Select(x, k)*:
  - Return the *k*-th smallest element in the subtree rooted at *x*
  - *BST-Select(T.root, k)* returns the *k*-th smallest elements in the entire tree.

- Using ideas just described, BST-Select(x, k) can be implemented to have $\Theta(height\ of\ tree)$ time complexity
  - which is $\Theta(\lg n)$ for a balanced binary tree.
  - See homework.

Are we done?

- Need to maintain the augmented information under dynamic changes of the tree!
  - i.e, under insertions / deletions
  - in this case, just adjusting this size count as we update nodes, or under rotations, and it does not increase asymptotic time complexity of these operations

- Remark:
  - Select() in a sorted array can be done in $\Theta(1)$ time.
  - However, an array does not support dynamic operations (insert/delete) efficiently. That's augmented BST is a better data structure in this case.

Summary

- Simple example of augmenting data structures
- In general, the augmented information can be quite complicated
  - Can be a separate data structure!
- Need to consider how to maintain such information under dynamic changes

FIN