DSC40B: Theoretical Foundations of Data Science II

Lecture 9: *Hashing and Hash table*

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Recall: (Dynamic) set operations

- Imagine you are maintaining a database indexed by some keys (real values), and you hope to support the following operations:
  - Search
  - Maximum
  - Minimum
  - Successor
  - Predecessor
  - Insert
  - Delete
  - Extract-Max
  - Increase-key

Using balanced BST, all these operations can be done in $O(\lg n)$ time

Dictionary operations

- Given a universe of elements U
  - these elements may not be numbers
- Need to store some keys
- Need to perform the following operations for keys
  - Search (membership queries)
  - Insert
  - Delete

We can use balanced BST for this purpose if keys can be compared.

But we can have even lighter weight data structure to handle these

Today

- Hashing in general
- Hash table
  - For dictionary operations

Part A:
The idea of Hashing

Hashing

- An important idea used commonly in practice

- Many uses:
  - Fast queries on a large data set.
  - Verifying message integrity.
  - Identify if file has changed in version control.

Hash function

- Mathematically, a hash function is simply a function
  - $f\colon\ U \to X$ from one set to another

- To make it useful, in practice,
  - It often maps some potentially large or complex object to a much smaller and simpler "fingerprint" or "signature"
  - One also wants the mapping to be "uniform" and cause few "collisions".
  - Note: a hash function needs to be deterministic!
    - Hashing the same object twice, we should get the same answer

Some examples

- A cryptographic hash function:
  - maps data of arbitrary size into an often fixed-sized output of much smaller size
  - e.g, MD5: maps it to a 128-bit value
  - hard to "reverse engineer" input from hash.
  - two similar input could and should lead to very different hash values

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  - two similar input could and should lead to very different hash values

```
> echo "My name is Justin" | md5
a741d8524a853cf83ca21eabf8cea190
> echo "My name is Justin!" | md5
f11eed2391bbd0a5a2355397c089fafd
```

```
> md5 slides.pdf
e3fd4370fda30ceb978390004e07b9df
```

A cryptographic hash function

- Why?
  - I release a piece of software.
  - I host it on Google Drive.
  - Someone (Google, US Gov., etc.) decides to insert extra code into software to spy on users.
  - A user has no way of knowing.

A cryptographic hash function

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- What do I do?
  - I release a piece of software and **publish the hash**
  - I host it on Google Drive.
  - Someone inserts extra code into software to spy on users.
  - You download the software and hash it. If hash is different, you know the file has been changed!

Some examples

- Want to place images into 100 bins.
  - How do we decide which bin an image goes into?
  - Hash function!
  - Takes in an image.
  - Outputs a number in {1, 2, …, 100}

Part B:
Hash Table

Dictionary operations

- Given a universe of elements U
- Need to store some keys
- Need to perform the following operations for keys
  - Insert
  - Search
  - Delete

- In other words, the key operation is membership queries (search), but also allows dynamic updates (insert, delete).

Approach 0

- Use an array to organize all the keys
  - We could pre-sort the array.
    - Preprocessing time:
  - Search:
  - Insert / Delete:

Approach 1

- Organize all keys in a doubly-linked list
  - Pre-processing: None
  - Insert:
  - Delete
  - Search:

Approach 2

- Organize all keys in a balanced BST
  - Search:
  - Insert:
  - Delete

Approach 3

- Direct address tables (DAT)

Example: Query for age

- Is there any student whose age is 17 ?

PID	Name	Age
A1843	Wan	24
A8293	Deveron	22
A9821	Vinod	41
A8172	Aleix	17
A2882	Kayden	4
A1829	Raghu	51
A9772	Cui	48
⋮	⋮	⋮

Approach 3

- Direct address tables (DAT)
- Suppose we know that all the keys we ever care are from 0 to $N = 150$
  - Like in the case when we query for ages,
- Open an array A of size N
  - If a given age z is in, set $A[z] \geq 1$, and 0 otherwise
  - Given a query age, say 22, simply return $A[22]$
  - Search:
  - Insert:
  - Delete:

What's the problem of DAT approach?

- DAT (directed address tables) very efficient

- However
  - Require the keys to be integers
  - Require the keys to be a fixed range
  - Require a size as large as this range – which makes it hard to use for a huge universe (of keys)

How about checking for names?

- Did the name Smith appear in the student database?

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Hash Table:
A way to use DAT idea, but allowing to use much more general universe through the "hash" map.

Use Hash Table!

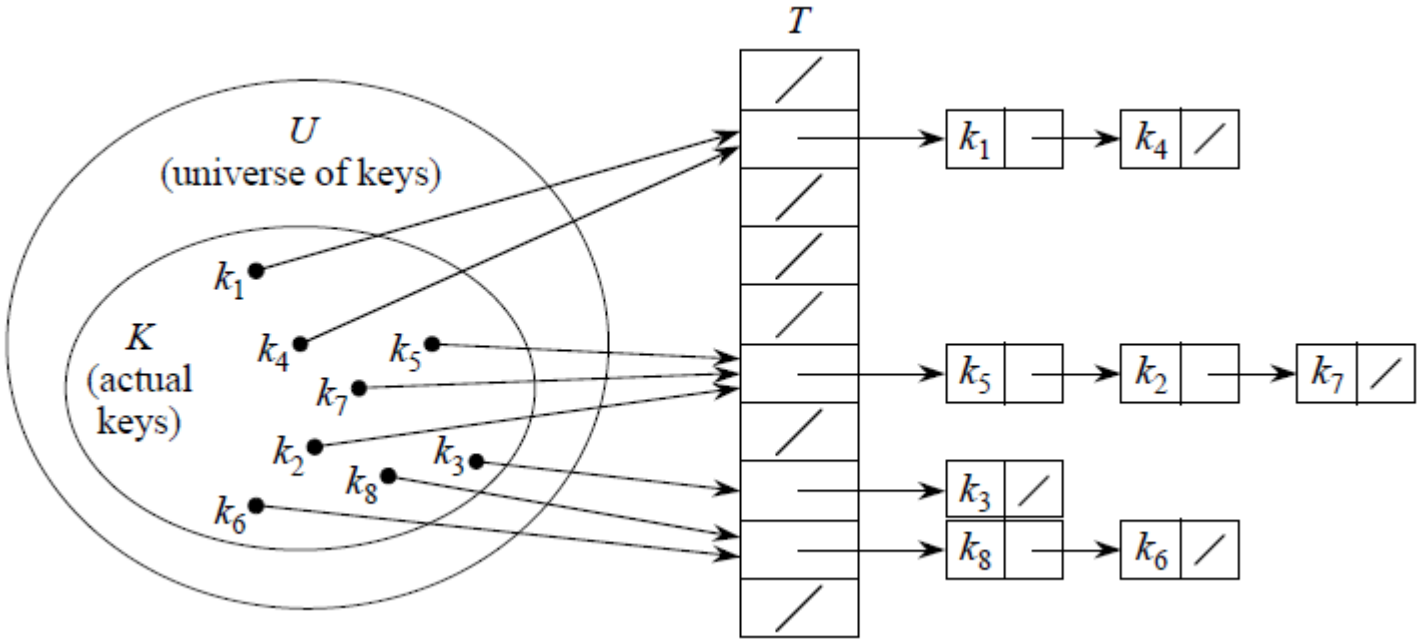
- U : universe
- $T[0 \ \ldots \ m-1]$: a hash table of size m
  - $m \ll |U|$
  - usually, we choose m to be around the size of data we expect to see
- Hash functions
  - $h: U \to \{0, 1, \ldots, m-1\}$
    - i.e, h maps each element in the universe to an index in the hash-table
- $h(k)$ is called the hash value of key *k*.
  - Given a key *k*, we will store it in location *h(k)* of hash table *T*,
    - i.e, store it at *T[h(k)]*

Collision

- Since the size of hash table is smaller than the universe:
  - Multiple keys may hash to the same slot.
  - A collision happens when $h(x) = h(y)$ for $x \neq y \in U$

- How to handle collisions?
  - Chaining
  - Open addressing
  - …

Collision Resolved by Chaining



- *T[j]*: a pointer to the head of the linked list of all stored elements that hash to *j*
- Nil otherwise

A simple example

- U: all positive integers $\mathbb{N}$
- Hash table $T[0,..,10]$
- Hash function: $h: \mathbb{N} \rightarrow [0, \ldots, 10]$
  - $h(x) = x \; mod \; 11$

Dictionary operations

- Chained-Hash-Insert (T, x)
  - Insert x at the head of list *T[h(x)]*
- Chained-Hash-Search(T, x)
  - Search for an element with key x in list *T[h(x)]*
- Chained-Hash-Delete(T, x)
  - Delete x from the list *T[h(x)]*

Dictionary operations

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  - Insert x at the head of list *T[h(x)]*

O(1)

- Chained-Hash-Search(T, x)
  - Search for an element with key x in list *T[h(x)]*

O(length(T[h(x)])

- Chained-Hash-Delete(T, x)
  - Delete x from the list *T[h(x)]*

O(length(T[h(x)])

Good Hash Function

- Performance of Hash table operations
  - depend on the number of elements hashed to each slot in hash table.

- Intuitively,
  - A good hash function should spread elements into the hash table uniformly
  - If there are n elements in the input data and m slots
    - then ideally there should be $\frac{n}{m}$ elements in each hash table slot.

Average case analysis

- n: # elements in the table
- m: size of table (# slots in the table)
- Load factor:
  - $\alpha = \frac{n}{m}$: average number of elements per linked list
  - Intuitively the optimal time needed
- Individual operation can be slow ($\Theta(n)$ worst case time)
- *Under certain assumption of the distribution of keys*, analyze expected performance.

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  - Time to find the correct bin:
  - Expected number of elements in this bin:
  - Time to perform linear search in this bin:
  - Total:

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- *Under certain assumption of the distribution of keys*, analyze expected performance. In particular, for a search query:
  - Time to find the correct bin: $\Theta(1)$
  - Expected number of elements in this bin: n/m
  - Time to perform linear search in this bin: $\Theta(n/m)$
  - Total: $\Theta(1 + n/m)$

Expected running time

- Under Simple uniform hashing assumption

- Search:
  - Expected time: $\mathrm{ET(n)} = \Theta(1 + \frac{n}{m})$
    - (worst case time $T(n) = \Theta(n)$)
- Insert:
  - $\mathrm{T(n)} = \Theta(1)$
- Delete:
  - Expected time: $\mathrm{ET(n)} = \Theta(1 + \frac{n}{m})$
    - (worst case time $T(n) = \Theta(n)$)

Key message:
if the load factor $\alpha = \frac{n}{m} = \Theta(1)$
then $\Theta(1)$ expected time for these operations!

In summary

- How fast can we query / insert with these data structures?

	Query	Insert
Unsorted linked list		
Unsorted array		
Sorted array		
BST		
Hash Table (expected time)		

Growing the Hash Table

- Insertions / search take $\Theta(1)$ expected time under the simple uniform hashing assumption, and that $\frac{n}{m} \leq c$ for some constant c

- We need to ensure that $n \leq c \cdot m$

- As we add more elements (thus n increases), we might need to increase m
  - -- need resizing strategies in hash table

Hashing in Python

- dict and set implement hash tables
  - set : stores only a set of values
  - dict : stores a set of (key, value) pairs, and query is on “key”

- Querying is done using in:

```
>>> # make a set
>>> L = {3, 6, -2, 1, 7, 12}
>>> 1 in L # Theta(1)
False
>>> 7 in L # Theta(1)
True
```

Part C:
Some examples of using hash tables and downsides of hash tables

Recall the movie problem

- The Movie problem
  - Input: Given a list of length of movies available, stored in array $movies$, and a flight duration D
  - Output: Return two movies whose total length = D; None otherwise.

Recall

- The naïve algorithm solves it in $\Theta(n^2)$ time
- But earlier, we mentioned we can do better by
  - First sorting the array of movie lengths
  - Then check for each movie *x*, whether there exits one whose length is *D – length(x)*
  - Worst case time complexity $T(n) = \Theta(n \lg n)$

New approach via Hashing

- Use Hash table, and frame this as a membership query problem

```
def optimize_entertainment_hash(times, D):
    hash_table = dict()
    for i, time in enumerate(times):
        hash_table[time] = i

    for i, time in enumerate(times):
        target = D - time
        if target in hash_table:
        return i, hash_table[target]
    return None
```

New approach via Hashing

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```

Expected time?
$\Theta(n)$

Another example

- Anagrams
  - Two strings w_1 and w_2 are anagrams if the letters of w_1 can be permuted to make w_2.
    - E.g, "eat" and "tea" , or "listen" and "silent"

- The *Anagrams problem*
  - Given a collection of n strings, determine if any two of them are anagrams.

Using Hash table

- Observation:
  - two strings are anagrams if their sorted lists are equal
    - sorted(w_1) == sorted(w_2)

```
def any_anagrams(words):
        seen = set()
        for word in words:
                w = sorted(word)
                if w in seen
                        return True
                else:
                        seen.add(w)
         return False
```

Using Hash table

- Observation:
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```

Expected time?
$\Theta(n)$

Hashing Downsides

- Only support dictionary queries
  - i.e, membership queries + insert / delete
  - Example 1: cannot be used to query for the two movies whose total time is closest to *D*
  - Example 2: cannot be used for performing a range query in a list of numbers
    - Say report the number of numbers fall within range [a, b]
  - Cannot be used for min / max / median / order statistics etc
    - BST wins when "order" type of query is important

Hashing Downsides

- Only support dictionary queries
  - i.e, membership queries + insert / delete

- No locality: similar items map to very different bins
  - Necessarily so for the performance of hashing in most cases!
  - But, in practice, we often query similar objects continuously
    - May result in many cache misses, slow

Summary

- Hashing
  - Very useful idea
  - Generates a small signature for a potentially large complex object

- Hash Table
  - Very practical data structure for dictionary operations
    - Very efficient for these operations, can be constant expected time if the load factor $\alpha = \frac{n}{m} = \Theta(1)$
  - Especially when the number of keys necessary is much smaller than the size of universe where input objects could come from!
  - In practice, need to choose hash functions properly
    - There exist intelligent hashing schemes

FIN