DSC40B:
Theoretical Foundations of Data Science II

Lecture 13: *Depth-First Search (DFS)*

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One more property about BFS

- For BFS algorithm, at any moment of the algorithm:
  - Recall the queue stores all the pending nodes (discovered, but not explored)
  - Recall we are inserting nodes into the queue in non-decreasing order their distance from source
  - It is easy to verify that
    - The shortest path distance from the source are non-decreasing in the queue
    - The shortest path distance for nodes in the queue cannot differ more than 1

Prelude

- Previously, Search strategy :
- How to decide which is the next node to explore?
  - BFS (Breadth-first search):
    - choose the ``oldest'' pending node to explore and expand
    - consequently, it explores as wide as possible before goes any ``deeper'' (in terms of distance to the source)
      - i.e, it visits all nodes at distance k to the source before moving to any node at distance $k+1$ from the source.

- Today: Depth-first search (DFS):
  - Choose the ``newest'' pending node to explore
  - Consequently, it will go as deep (farther away from source) as possible during exploration

DFS algorithm and time complexity

Depth-first Search (DFS)

- DFS(G, s)
  - It will perform depth-first search in G starting from a graph node s called the *source* node.
  - At each step of exploration, take newest pending node to explore.

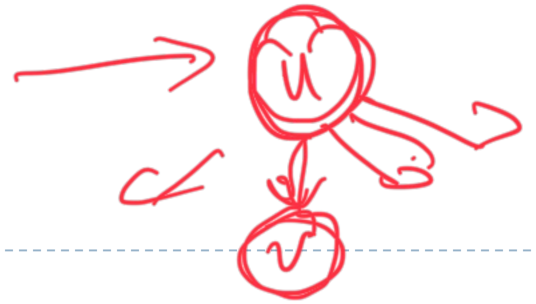
- To be able to extract the "newest" pending node
  - we need a standard stack data structure to provide FILO (first-in-last-out)

- In algorithm implementation,
  - we use recursive algorithm to achieve this FILO/stack idea implicitly

Implementation in Python

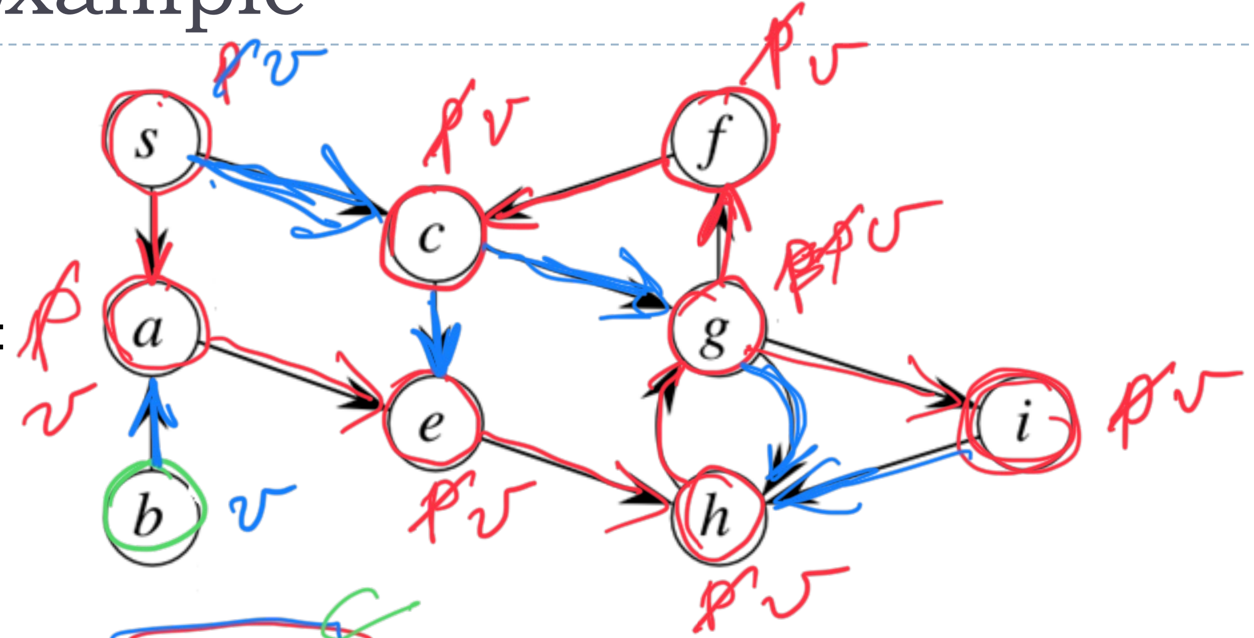
```python
def dfs(graph, u, status=None):
    """Start a DFS at `u`."""
    # initialize status if it was not passed
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}

    status[u] = 'pending'
    for v in graph.neighbors(u): # explore edge (u, v)
        if status[v] == 'undiscovered':
            dfs(graph, v, status)
    status[u] = 'visited'
```



Example

G:



- Call dfs(G, s)

dfs(G, a)

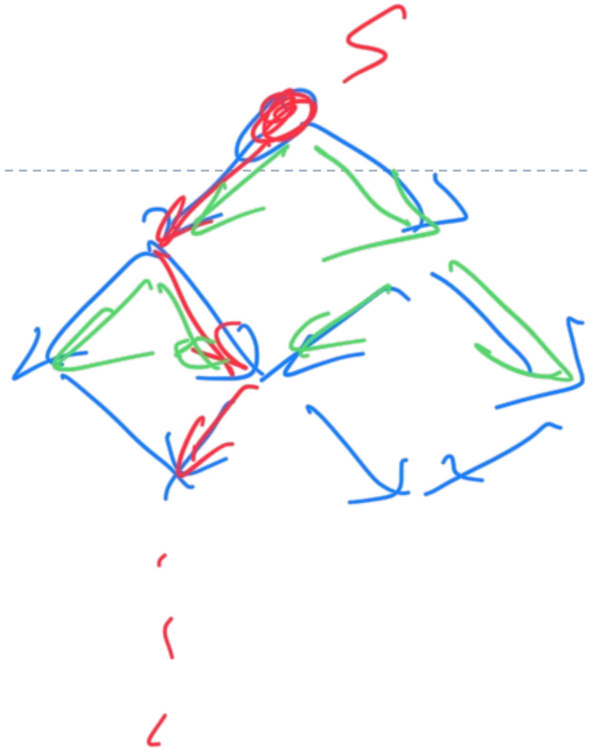
dfs(G, e)

dfs(G, h)

→ dfs(G, g)

dfs(f)

dfs(c)



dfs(b)

dfs(i)

Full DFS

- DFS will visit all nodes reachable from the source node
  - If the input graph is connected, then it will visit all nodes in the same connected component as the source
- To visit all nodes in a graph
  - Needs full-DFS
    - which requires we restart from undiscovered nodes.

```
def full_dfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered'
            dfs(graph, node, status)
```

Complete code

```python
def full_dfs(graph):
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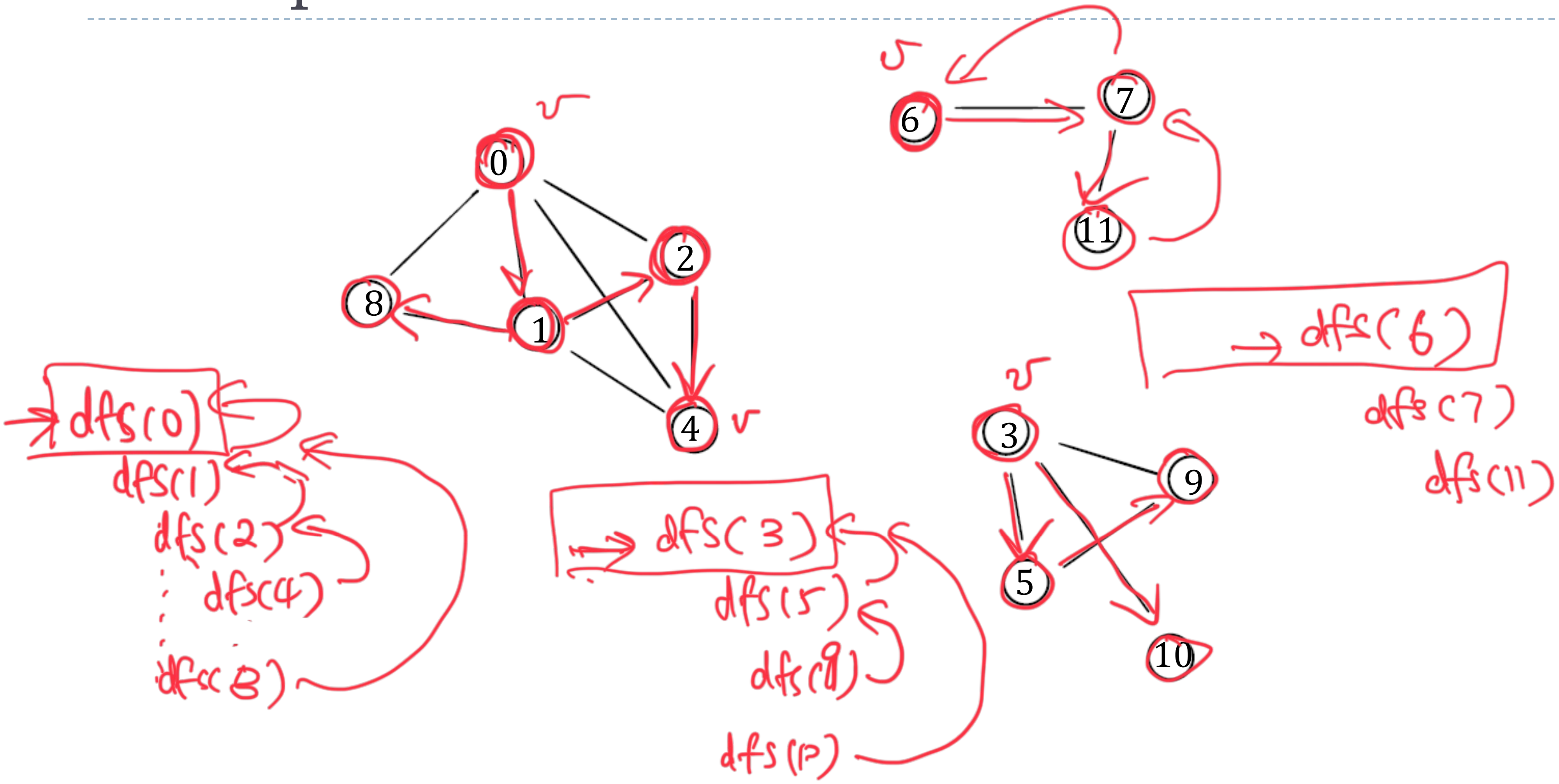
Example


0
1
2
4
8
6
7
11
3
5
9
10
dfs(0)
dfs(1)
dfs(2)
dfs(4)
dfs(8)
dfs(3)
dfs(5)
dfs(9)
dfs(10)
dfs(6)
dfs(7)
dfs(11)

Time complexity analysis

- (similar to BFS): for full-DFS
  - Each node will be explored exactly once
    - Each edge will be traversed (visited) twice for undirected graphs
    - Each edge will be traversed (visited) once will for directed graphs
  - Hence total time complexity of full-DFS
    - $\Theta(|V| + |E|)$

DFS Tree, start and finish times

full_DFS

```
def full_dfs(graph):
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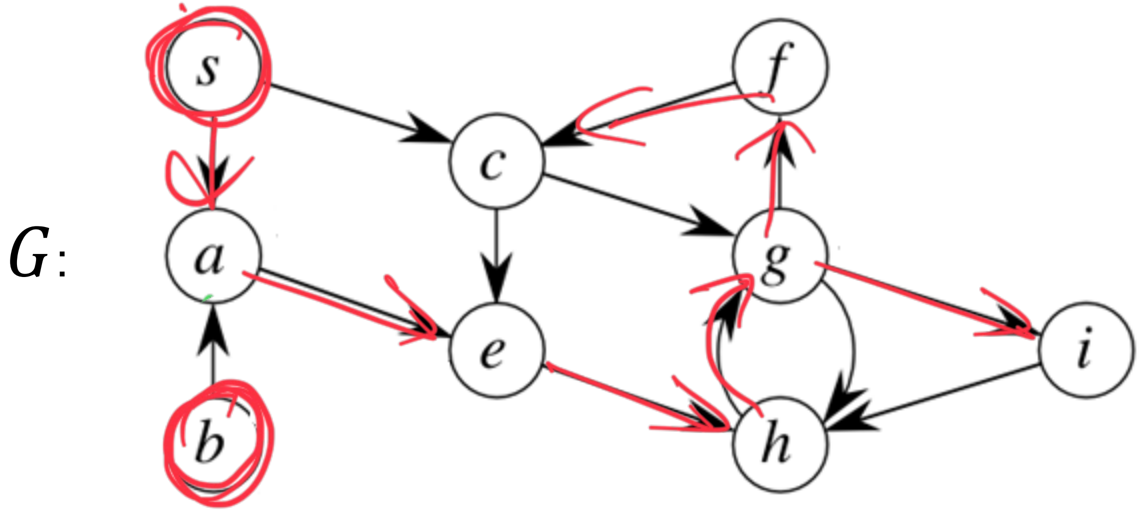
    status[u] = 'visited'
```

"Parent" (Predecessor) information

- Similar to BFS, for each node v,
  - its (DFS-)predecessor is the node u where through exploring edge (u, v) the node v was first discovered (status changed to pending).
- Collection of edges of the form (predecessor(v), v) will give a tree
  - called DFS-tree.
  - Predecessor(v) is the parent of v in this DFS-tree.

Observations

- Between marking a node as pending and visited, many other nodes are marked pending or visited.



Start and Finish times

- A node status changed
  - from undiscovered to pending:
    - first time this node is discovered
  - from pending to visited:
    - exploration of this node is finished
- Keep an integer running clock
- For each node:
  - Start time: status changes from undiscovered to pending
  - Finish time: status changes from pending to visited
- Clock increments by 1 whenever some node is marked pending/visiting

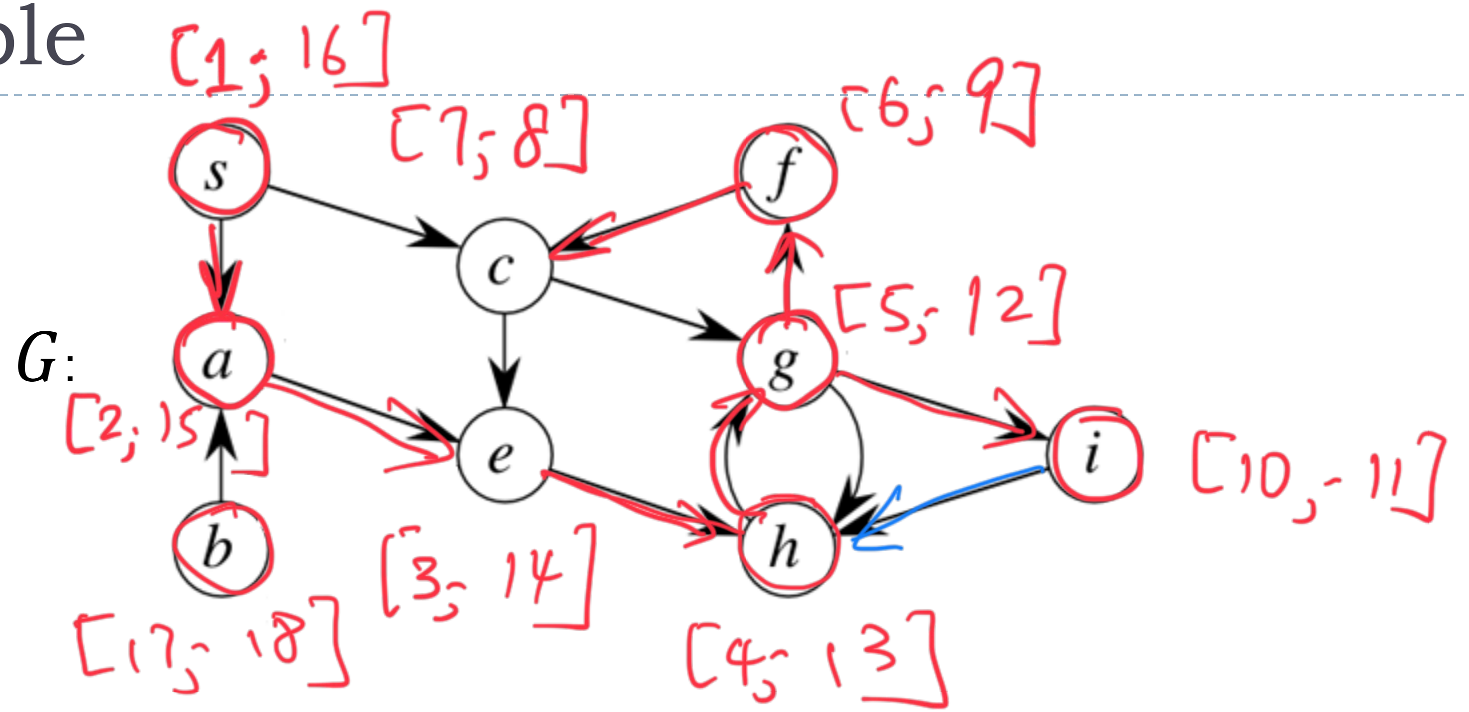
full_DFS_times implementation

```python
from dataclasses import dataclass
@dataclass
class Times:
    clock: int
    start: dict
    finish: dict
```

```python
def full_dfs_times(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    predecessor = {node: None for node in graph.nodes}
    times = Times(clock=0, start={}, finish={})
    for u in graph.nodes:
        if status[u] == 'undiscovered':
            dfs_times(graph, u, status, times)
    return times, predecessor
```

```python
def dfs_times(graph, u, status, predecessor, times):
    times.clock += 1
    times.start[u] = times.clock
    status[u] = 'pending'
    for v in graph.neighbors(u): # explore edge (u, v)
        if status[v] == 'undiscovered':
            predecessor[v] = u
            dfs_times(graph, v, status, times)
    status[u] = 'visited'
    times.clock += 1
    times.finish[u] = times.clock
```

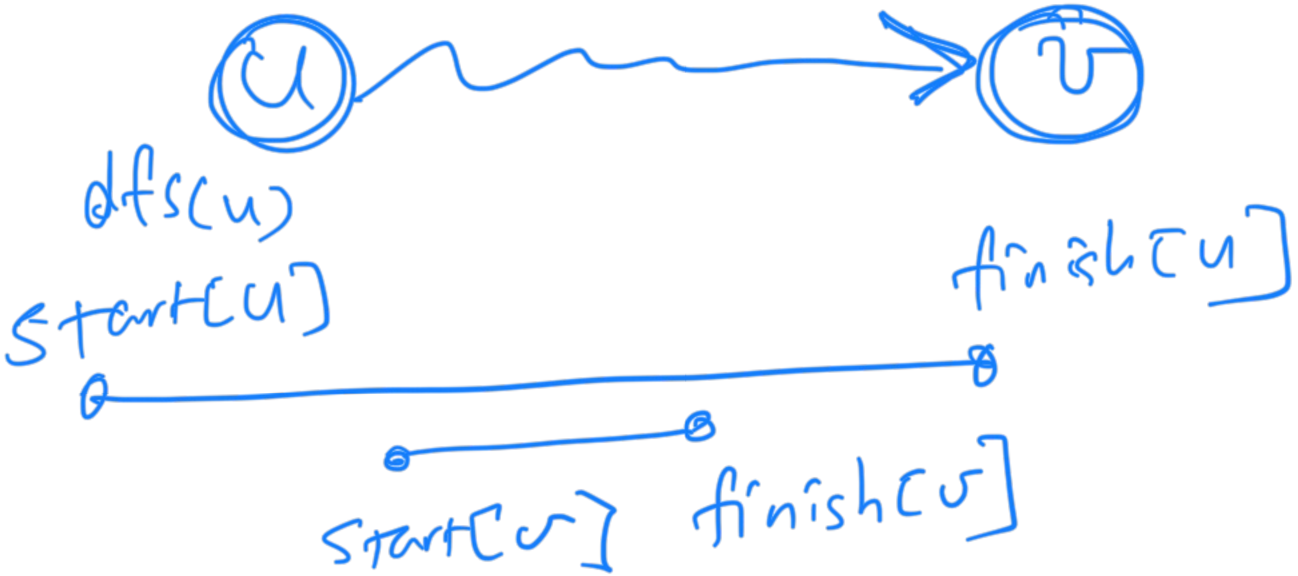
Example

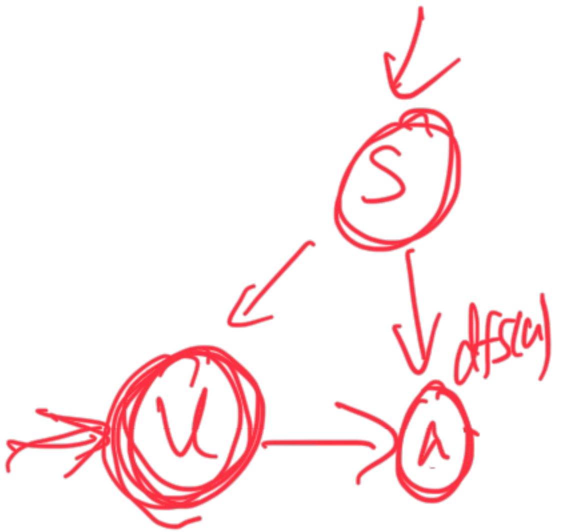


Nesting properties in DFS

DFS Key Property 1

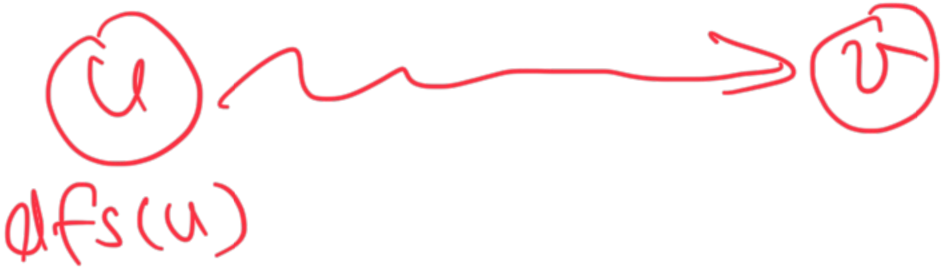
- Take any two nodes u and v where v is reachable from u
  - If while exploring u we reached v, then the exploration of v has to be done first before we finish exploring u
    - That is: start[u] $\leq$ start[v] $\leq$ finish[v] $\leq$ finish[u]





Exercise

True or **False**: if v is reachable from u and v is undiscovered when `dfs(u)` is called, then `dfs(v)` must be called during `dfs(u)`.



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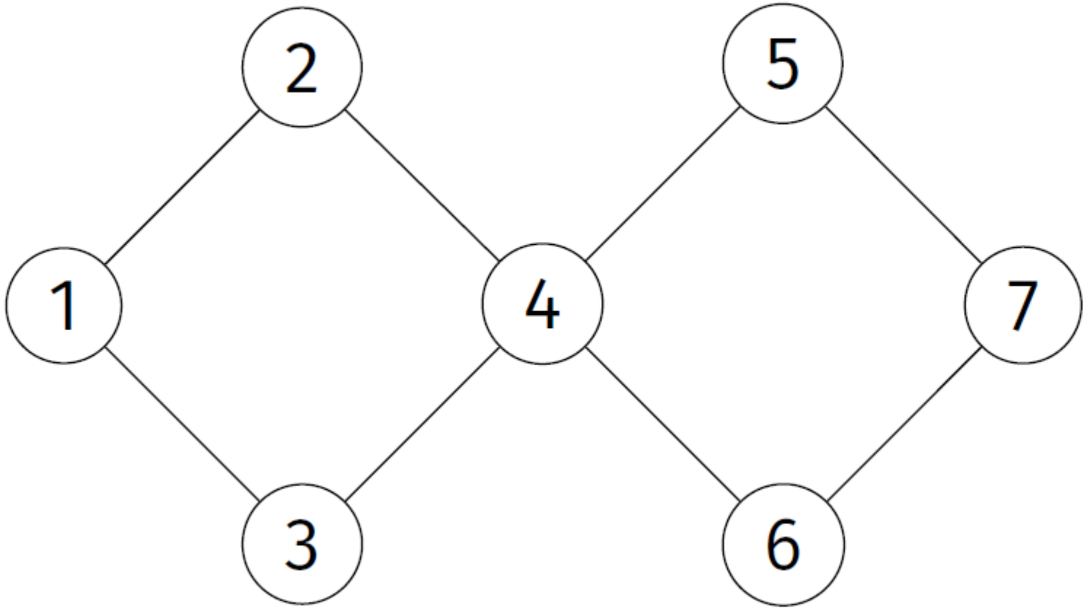
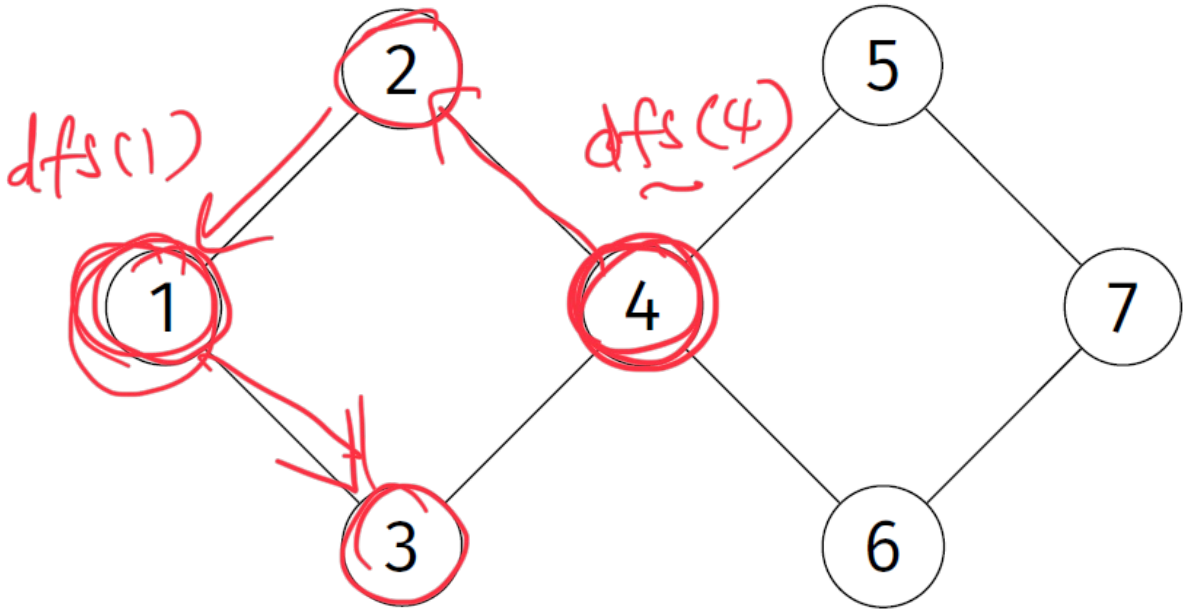


dfs(4)
dfs(2)
dfs(1)

Exercise

True or **False**: if v is reachable from u and v is undiscovered when `dfs(u)` is called, then `dfs(v)` must be called during `dfs(u)`.

- False!
  - Suppose dfs(4) is the root call.
  - When dfs(1) is called, node 5 is undiscovered
  - But dfs(5) is not called during dfs(1)



DFS Key Property 2

- If at time dfs(u) is called
  - If v is undiscovered, and
  - There is a path of undiscovered nodes from u to v
- Then
  - dfs(v) will start and finish during the call of dfs(u)

DFS Key Property 2

- If at time dfs(u) is called
  - If v is undiscovered, and
  - There is a path of undiscovered nodes from u to v
- Then
  - dfs(v) will start and finish during the call of dfs(u)

- In fact, during the call of dfs(u):
  - It will visit exactly those nodes v who is reachable from u by a path of undiscovered at the time.

Exercise

Suppose while visiting node u, we see that neighbor v is **pending**. True or False: there is a path from v to u.

Exercise

Suppose while visiting node u, we see that neighbor v is **pending**. True or False: there is a path from v to u.

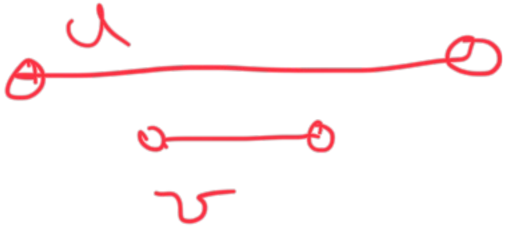
- True!

Nesting Property of DFS

- Claim A:
  - Take any two nodes u and v. Assume start[u] $\leq$ start[v]
  - Exactly one of the following two is true:
    - start[u] $\leq$ start[v] $\leq$ finish[v] $\leq$ finish[u]
    - start[u] $\leq$ finish[u] $\leq$ start[v] $\leq$ finish[v]

start[u] finish[u]

start[v] finish[v]



Nesting Property of DFS

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- In particular, if when dfs(u) is called there is a path of undiscovered nodes from u to v, then
  - start$[u] \leq$ start$[v] \leq$ finish$[v] \leq$ finish$[u]$
- Otherwise:
  - start$[u] \leq$ finish$[u] \leq$ start$[v] \leq$ finish$[v]$

DAGs and topological sort

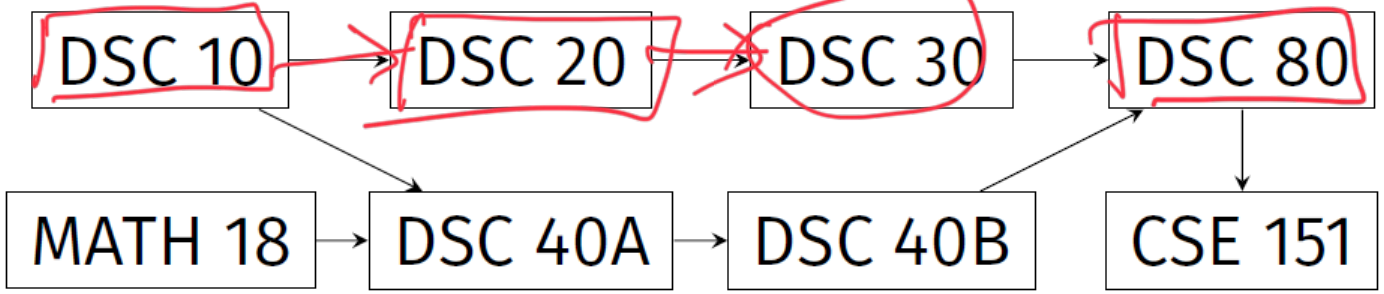
Applications of DFS

- Is node v reachable from u?
- Is a undirected graph connected?
- How many connected components are there in a undirected graph?
- Is the input graph a tree?
- Find the shortest path to a source node u?
  - NO !
  - Unlike BFS.
- But DFS can be used for sth. that BFS cannot provide
  - Topological sort for directed acyclic graphs (DAGs)

10, 20, 30, 80, 18,

Prerequisite graphs

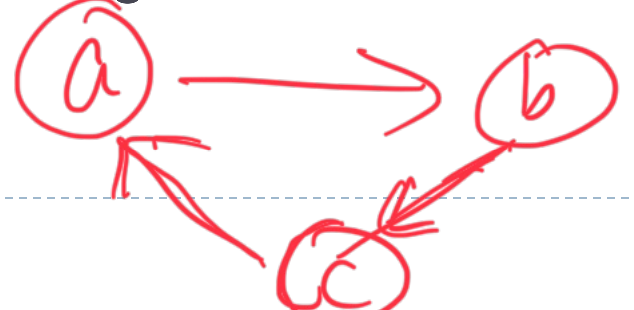
40A, 40B, 151



10, 20, 40A, 40B, 80

30

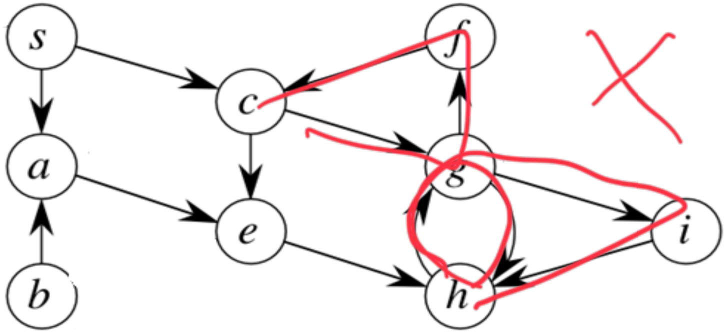
- Note that this graph is a directed graph
  - edge (u, v) means that course u is prerequisite for v
- Goal:
  - Find an ordering so as to take these classes satisfying all prerequisite requirements

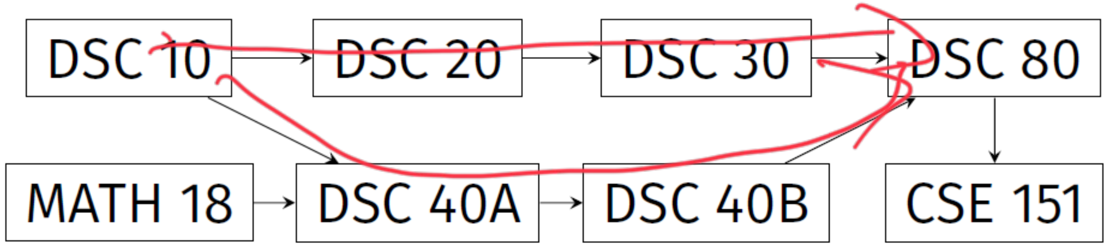


Directed Acyclic Graphs (DAGs)

- Recall: A cycle is a path from a node to itself.
- A directed acyclic graph (DAG) is a directed graph that does not contain any (directed) cycles

G:





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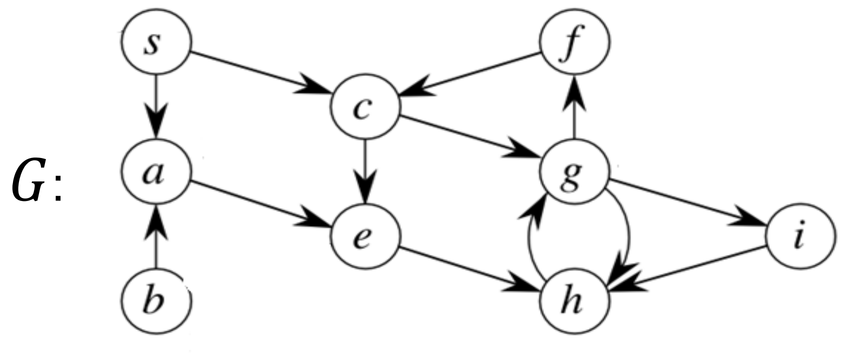
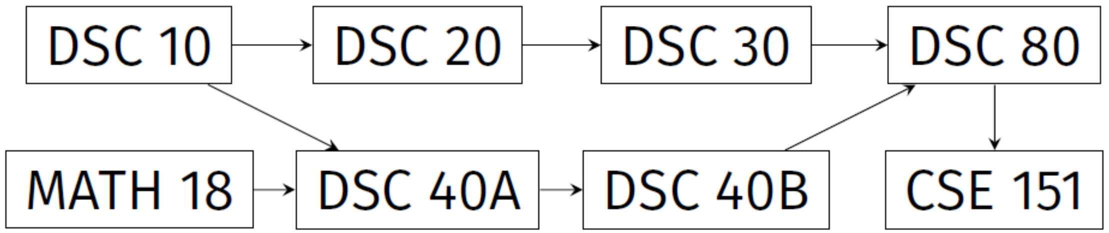


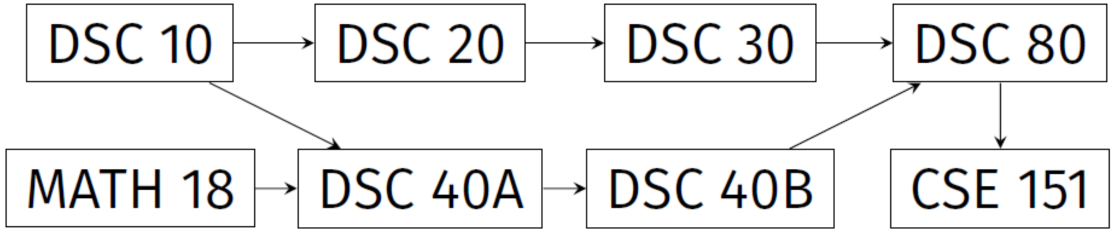
Not a DAG!



A DAG!

Topological Sorts

- Given a DAG $G = (V, E)$, a topological sort of G is an ordering of V such that
  - for any edge $(u, v) \in E$, then u comes before v in this ordering



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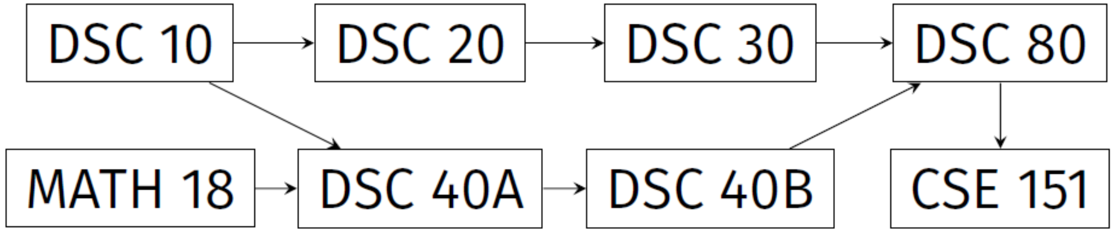

A topological sort:
DSC10, Math18, DSC20, DSC40A, DSC30, DSC40B, DSC80, DSC151

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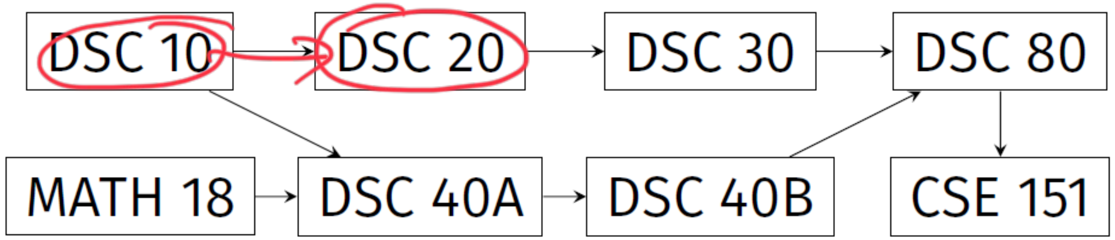
a ⟶ b ⟶ c ⟶ . . ⟶



A topological sort:
DSC10, Math18, DSC20, DSC40A, DSC30, DSC40B, DSC80, DSC151

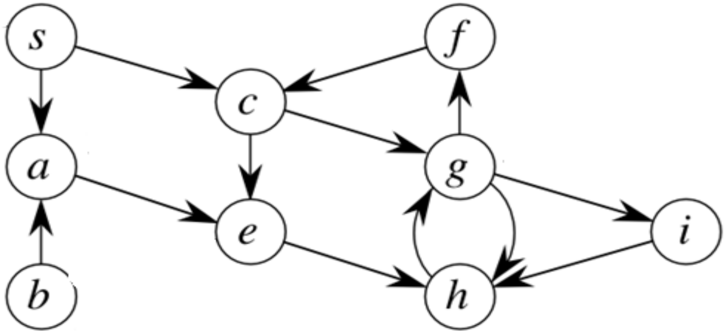
Another topological sort:
Math18, DSC10, DSC20, DSC30, DSC40A, DSC40B, DSC80, DSC151

- Topological sorts of the same DAG are not unique.

Why DAG?

- Claim:
  - A directed graph $G = (V, E)$ can admit a topological sort if and only if G is a DAG!

G:

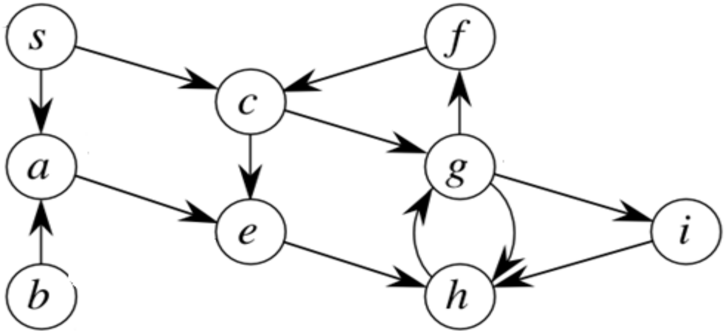


Why DAG?

- Claim:
  - A directed graph $G = (V, E)$ can admit a topological sort if and only if G is a DAG!

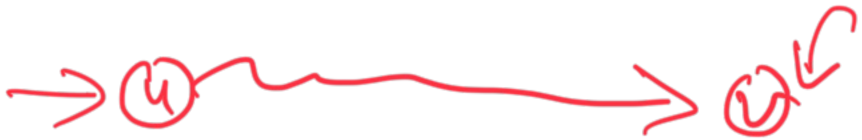
- Why?
  - If there is a cycle, then there is no valid ordering for nodes in that cycle!
  - If it is a DAG, then we will give an algorithm to show we can compute topological sort.

G:



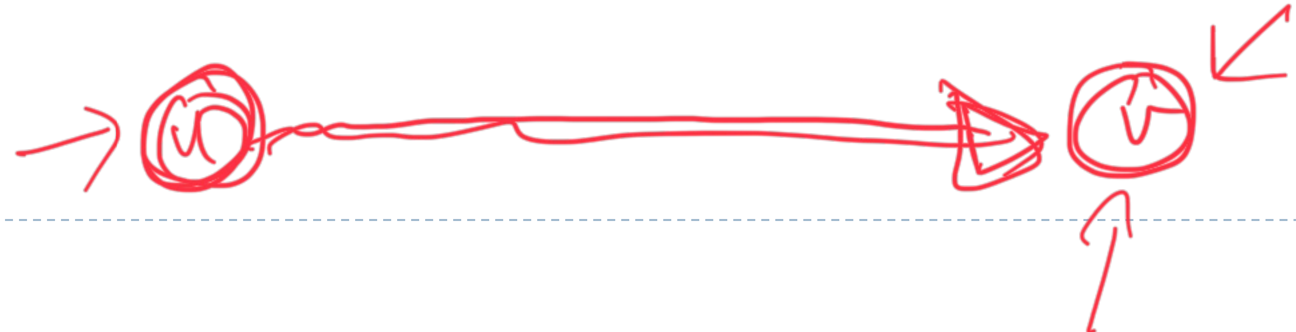
- Given a DAG $G = (V, E)$, we aim to have an algorithm to compute a topological sort of G

DFS Key Property 3



- Recall that for any two nodes u and v where v is reachable from u
  - If while exploring u we reached v, then the exploration of v has to be done first before we finish exploring u
    - That is: start[u] $\leq$ start[v] $\leq$ finish[v] $\leq$ finish[u]

- Claim B: If a directed graph $G = (V, E)$ does not have cycles, and node v is reachable from u, then finish[v] $\leq$ finish[u]



An algorithm to compute topo-sort

- Suppose we are given a DAG G.
- Consider any two nodes u, v such that v is reachable from u.
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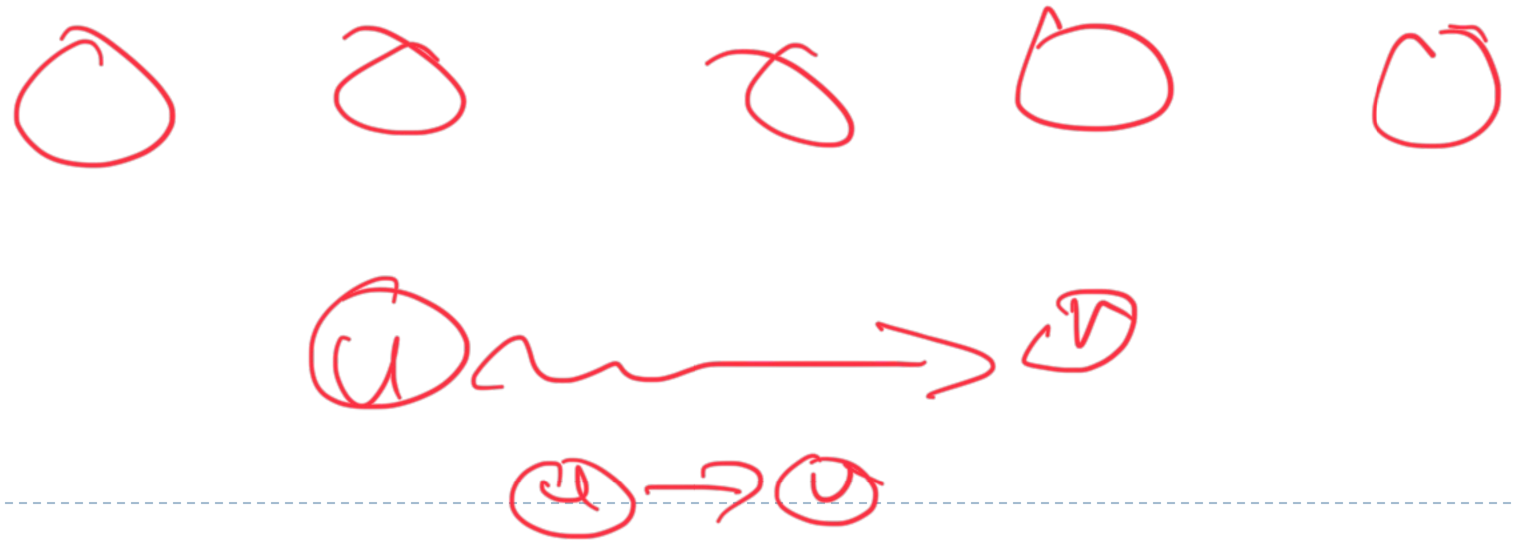
- Hence nodes with later finish-time should come first in any topological sort!



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- Topo-sort Algorithm:
  - First perform DFS on input graph $G = (V, E)$ → $\Theta(V+E)$
  - Output the order in decreasing order of finish-time. → $\Theta(V)$

$G = (V, E)$

Time Complexity of Topo-Sort Algorithm

- Topo-sort Algorithm:
  - First perform DFS on input graph $G = (V, E)$
  - Output the order in decreasing order of finish-time.

- In general, sorting $|V|$ numbers takes $\Theta(|V| \log |V|)$ time.

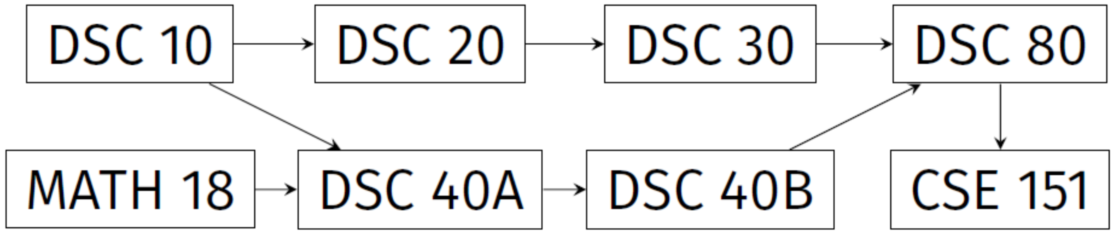
Time Complexity of Topo-Sort Algorithm

- Topo-sort Algorithm:
  - First perform DFS on input graph $G = (V, E)$
  - Output the order in decreasing order of finish-time.

- In general, sorting $|V|$ numbers takes $\Theta(|V| \log |V|)$ time.
- However, here, all "finish-times" are integers between 1 to $2|V|$
  - Sorting them can be done in $\Theta(|V|)$ time!

- Total time complexity: $\Theta(|V| + |E|)$

Example



Remark:

- There are many other ways to compute a topological sort for a DAG in the same time complexity, without using DFS.

Summary

- DFS:
  - Yet another graph search strategy
  - Similarly to BFS, as a graph search strategy, can help solve many problems, such as checking for connectivity, reachability, finding a path and so on.

  - But has some different properties as BFS
    - BFS: useful for finding shortest path to the source
    - DFS: has nesting properties in terms of start/finish times.
      - An application: Topological sort in DAG

FIN