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DAWN

History records the names of royal bastards, but cannot tell us the origin of wheat.

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(1823-1915)

A million years or so have passed since the tool-wielding animal called man made its appearance on this planet. During this time it learned to recognize shapes and directions; to grasp the concepts of magnitude and number; to measure; and to realize that there exist relationships between certain magnitudes.

The details of this process are unknown. The first dim flash in the darkness goes back to the stone age — the bone of a wolf with incisions to form a tally stick (see figure on next page). The flashes become brighter and more numerous as time goes on, but not until about 2,000 B.C. do the hard facts start to emerge by direct documentation rather than by circumstantial evidence. And one of these facts is this: By 2,000 B.C., men had grasped the significance of the constant that is today denoted by π, and that they had found a rough approximation of its value.

How had they arrived at this point? To answer this question, we must return into the stone age and beyond, and into the realm of speculation.

Long before the invention of the wheel, man must have learned to identify the peculiarly regular shape of the circle. He saw it in the pupils of his fellow men and fellow animals; he saw it bounding the disks of the Moon and Sun; he saw it, or something near it, in some flowers; and perhaps he was pleased by its infinite symmetry as he drew its shape in the sand with a stick.

Then, one might speculate, men began to grasp the concept of magnitude — there were large circles and small circles, tall trees and small trees, heavy stones, heavier stones, very heavy stones. The transition from these qualitative statements to quantitative measure-

ment was the dawn of mathematics. It must have been a long and arduous road, but it is a safe guess that it was first taken for quantities that assume only integral values — people, animals, trees, stones, sticks. For counting is a quantitative measurement: The measurement of the amount of a multitude of items.

Man first learned to count to two, and a long time elapsed before he learned to count to higher numbers. There is a fair amount of evidence for this,[1] perhaps none of it more fascinating than that preserved in man's languages: In Czech, until the Middle Ages, there used to be two kinds of plural — one for two items, another for many (more than two) items, and apparently in Finnish this is so to this day. There is evidently no connection between the (Germanic) words *two* and *half*; there is none in the Romance languages (French: *deux* and *moitié*) nor in the Slavic languages (Russian: *dva* and *pol*), and in Hungarian, which is not an Indo-European language, the words are *kettő* and *fél*. Yet in all European languages, the words for 3 and 1/3, 4 and 1/4, etc., are related. This suggests that men grasped the concept of a ratio, and the idea of a relation between a number and its reciprocal, only after they had learned to count beyond two.

A stone age tally stick. The tibia (shin) of a wolf with two long incisions in the center, and two series of 25 and 30 marks. Found in Věstonice, Moravia (Czechoslovakia) in 1937.[2]

The next step was to discover relations between various magnitudes. Again, it seems certain that such relations were first expressed qualitatively. It must have been noticed that bigger stones are heavier, or to put it into more complicated words, that there is a relation between the volume and the weight of a stone. It must have been observed that an older tree is taller, that a faster runner covers a longer distance, that more prey gives more food, that larger fields yield bigger crops. Among all these kinds of relationships, there was one which could hardly have escaped notice, and which, moreover, had no exceptions:

The wider a circle is "across," the longer it is "around."

And again, this line of qualitative reasoning must have been followed by quantitative considerations. If the volume of a stone is doubled, the weight is doubled; if you run twice as fast, you cover double the distance; if you treble the fields, you treble the crop; if you double the diameter of a circle, you double its circumference. Of course, the rule does not always work: A tree twice as old is not twice as tall. The reason is that "the more . . . the more" does not always imply proportionality; or in more snobbish words, not every monotonic function is linear.

Neolithic man was hardly concerned with monotonic functions; but it is certain that men learned to recognize, consciously or unconsciously, by experience, instinct, reasoning, or all of these, the concept of proportionality; that is, they learned to recognize pairs of magnitude such that if the one was doubled, trebled, quadrupled, halved or left alone, then the other would also double, treble, quadruple, halve or show no change.

And then came the great discovery. By recognizing certain specific properties, and by defining them, little is accomplished. (That is why the old type of descriptive biology was so barren.) But a great scientific discovery has been made when the observations are generalized in such a way that a generally valid rule can be stated. The greater its range of validity, the greater its significance. To say that one field will feed half the tribe, two fields will field the whole tribe, three fields will feed one and a half tribes, all this applies only to certain fields and tribes. To say that one bee has six legs, three bees have eighteen legs, etc., is a statement that applies, at best, to the class of insects.But somewhere along the line some inquisitive and smart individuals must have seen something in common in the behavior of the magnitudes in these and similar statements:

No matter how the two proportional quantities are varied, their ratio remains constant.

For the fields, this constant is $1 : \frac{1}{2} = 2 : 1 = 3 : 1\frac{1}{2} = 2$. For the bees, this constant is $1 : 6 = 3 : 18 = 1/6$. And thus, man had discovered a general, not a specific, truth.

This constant ratio was not obtained by numerical division (and certainly not by the use of Arabic numerals, as above); more likely, the ratio was expressed geometrically, for geometry was the first mathematical discipline to make substantial progress. But the actual technique of arriving at the constancy of the ratio of two proportional quantities makes little difference to the argument.

There were of course many intermediate steps, such as the discovery of sums, differences, products and ratios; and the step of abstraction, exemplified by the transition from the statement "two birds and two birds make four birds" to the statement "two and two is four." But the decisive and great step on the road to π was the discovery that proportional quantities have a constant ratio.

From here it was but a dwarf's step to the constant π: If the "around" (circumference) and the "across" (diameter) of a circle were recognized as proportional quantities, as they easily must have been, then it immediately follows that the ratio

circumference : diameter = constant for all circles.

This constant circle ratio was not denoted by the symbol π until the 18th century (A.D.), nor, for that matter, did the equal sign (=) come into general use before the 16th century A.D. (The twin lines as an equal sign were used by the English physician and mathematician Robert Recorde in 1557 with the charming explanation that "noe .2. thynges, can be moare equalle.") However, we shall use modern notation from the outset, so that the definition of the number π reads

$$\pi = \frac{C}{D} \tag{1}$$

where C is the circumference, and D the diameter of any circle.

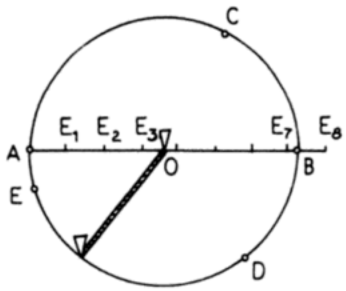
And with this, our speculative road has reached, about 2,000 B.C., the dawn of the documented history of mathematics. From the documents of that time it is evident that by then the Babylonians and the Egyptians (at least) were aware of the existence and significance of the constant π as given by (1).

BUT the Babylonians and the Egyptians knew more about π than its mere existence. They had also found its approximate value. By about 2,000 B.C., the Babylonians had arrived at the value

$$\pi = 3\,{}^1\!/_8 \tag{2}$$

and the Egyptians at the value

$$\pi = 4(8/9)^2 \tag{3}$$



How to measure π in the sands of the Nile

How did these ancient people arrive at these values? Nobody knows for certain, but this time the guessing is fairly easy.

Obviously, the easiest way is to take a circle, to measure its circumference and diameter, and to find π as the ratio of the two. Let us try to do just that, imagining that we are in Egypt in 3,000 B.C. There is no National Bureau of Standards; no calibrated measuring tapes. We are not allowed to use the decimal system or numerical division of any kind. No compasses, no pencil, no paper; all we have is stakes, ropes and sand.

So we find a fairly flat patch of wet sand along the Nile, drive in a stake, attach a piece of rope to it by loop and knot, tie the other end to another stake with a sharp point, and keeping the rope taut, we draw a circle in the sand. We pull out the central stake, leaving a hole O (see figure above). Now we take a longer piece of rope, choose any point A on the circle and stretch the rope from A across the hole O until it intersects the circle at B. We mark the length AB on the rope (with charcoal); this is the diameter of the circle and our unit of length. Now we take the rope and lay it into the circular groove in the sand, starting at A. The charcoal mark is at C; we have laid off the diameter along the circumference once. Then we lay it off a second time from C to D, and a third time from D to A, so that the diameter goes into the circumference three (plus a little bit) times.

If, to start with, we neglect the little bit, we have, to the nearest integer,

$$\pi = 3 \tag{3}$$

To improve our approximation, we next measure the little left-over bit EA as a fraction of our unit distance AB. We measure the curved

23 וַיַּעַשׂ אֶת־הַיָּם מוּצָק עֶשֶׂר
בָּאַמָּה מִשְּׂפָתוֹ עַד־שְׂפָתוֹ עָגֹל ׀ סָבִיב וְחָמֵשׁ בָּאַמָּה
קוֹמָתוֹ וקוה שְׁלֹשִׁים בָּאַמָּה יָסֹב אֹתוֹ סָבִיב׃

23. και εποιησε την θαλασσαν δεκα εν πηχει απο του χειλους αυτης εως του ψειλους αυτης, στρογγυλον κυελω το αυτο. πεντε εν πηχει το υψος αυτης. και συνηγμενη τρεις και τριακοντα εν πιχει.

[23] Hizo asimismo un mar de fundición, de diez codos del uno al otro lado, redondo, y de cinco codos de alto, y ceñíalo en derredor un cordón de treinta codos.

23. Il fit aussi une mer de fonte, de dix coudées d'un bord jusqu'à l'autre, qui était toute ronde: elle avait cinq coudées de haut, et elle était environnée tout à l'entour d'un cordon de trente coudées.

23. Udělal též moře slité, desíti loket od jednoho kraje k druhému, okrouhlé vůkol, a pět loket byla vysokost jeho, a okolek jeho třicet loket vůkol.

23. Und er machte ein Meer, gegossen, von einem Rand zum andern zehn Ellen weit, rundumher, und fünf Ellen hoch, und eine Schnur dreißig Ellen lang war das Maß ringsum.

23. And he made a molten sea, ten cubits from the one brim to the other; it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.

length EA and mark it on a piece of rope. Then we straighten the rope and lay it off along AB as many times as it will go. It will go into our unit distance AB between 7 and 8 times. (Actually, if we swindle a little and check by 20th century arithmetic, we find that 7 is much nearer the right value than 8, i.e., that E_7 in the figure on p. 13 is nearer to B than E_8, for $1/7 = 0.142857\ldots$, $1/8 = 0.125$, and the former value is nearer $\pi - 3 = 0.141592\ldots$ However, that would be difficult to ascertain by our measurement using thick, elastic ropes with coarse charcoal marks for the roughly circular curve in the sand whose surface was judged "flat" by arbitrary opinion.)

We have thus measured the length of the arc EA to be between 1/7 and 1/8 of the unit distance AB; and our second approximation is therefore

$$3\tfrac{1}{8} < \pi < 3\tfrac{1}{7} \tag{4}$$

for this, to the nearest simple fractions, is how often the unit rope length AB goes into the circumference $ABCD$.

And indeed, the values

$$\pi = 3, \qquad \pi = 3\tfrac{1}{7}, \qquad \pi = 3\tfrac{1}{8}$$

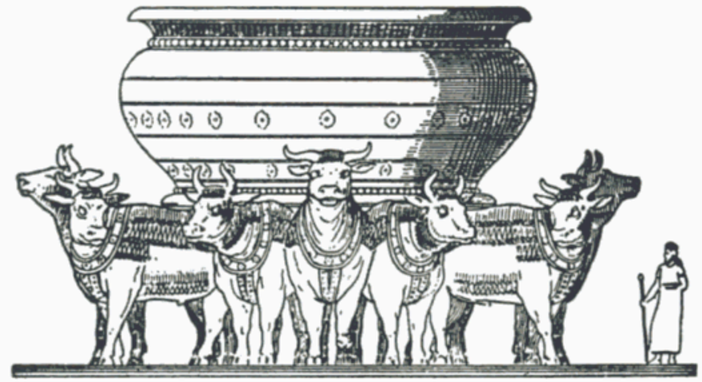
are the values most often met in antiquity.

For example, in the Old Testament (I Kings vii.23, and 2 Chronicles iv.2), we find the following verse:

> *"Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof: and a line of thirty cubits did compass it round about."*

The molten sea, we are told, is round; it measures 30 cubits round about (in circumference) and 10 cubits from brim to brim (in diameter); thus the biblical value of π is $30/10 = 3$.

The Book of Kings was edited by the ancient Jews as a religious work about 550 B.C., but its sources date back several centuries. At that time, π was already known to a considerably better accuracy, but evidently not to the editors of the Bible. The Jewish Talmud, which is essentially a commentary on the Old Testament, was published about 500 A.D. Even at this late date it also states "that which in circumference is three hands broad is one hand broad."



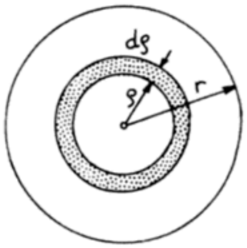
The molten sea as reconstructed by Gressman
from the description in 2 Kings vii.[2]

In early antiquity, in Egypt and other places, the priests were often closely connected with mathematics (as custodians of the calendar, and for other reasons to be discussed later). But as the process of specialization in society continued, science and religion drifted apart. By the time the Old Testament was edited, the two were already separated. The inaccuracy of the biblical value of π is, of course, no more than an amusing curiosity. Nevertheless, with the hindsight of what happened afterwards, it is interesting to note this little pebble on the road to confrontation between science and religion, which on several occasions broke out into open conflict, and about which we shall have more to say later.

Returning to the determination of π by direct measurement using primitive equipment, it can probably safely be said that it led to values no better than (4).

From now on, man had to rely on his wits rather than on ropes and stakes in the sand. And it was by his wits, rather than by experimental measurement, that he found the circle's area.

THE ancient peoples had rules for calculating the area of a circle. Again, we do not know how they derived them (except for one method used in Egypt, to be described in the next chapter), and once



Calculation of the area of a circle by integral calculus. The area of an elementary ring is $dA = 2\pi\rho d\rho$; hence the area of the circle is

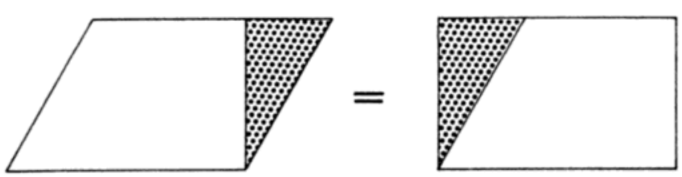
$$A = 2\pi \int_0^r \rho \, d\rho = \pi r^2.$$

more we have to play the game "How do you do it with their knowledge" to make a guess. The area of a circle, we know, is

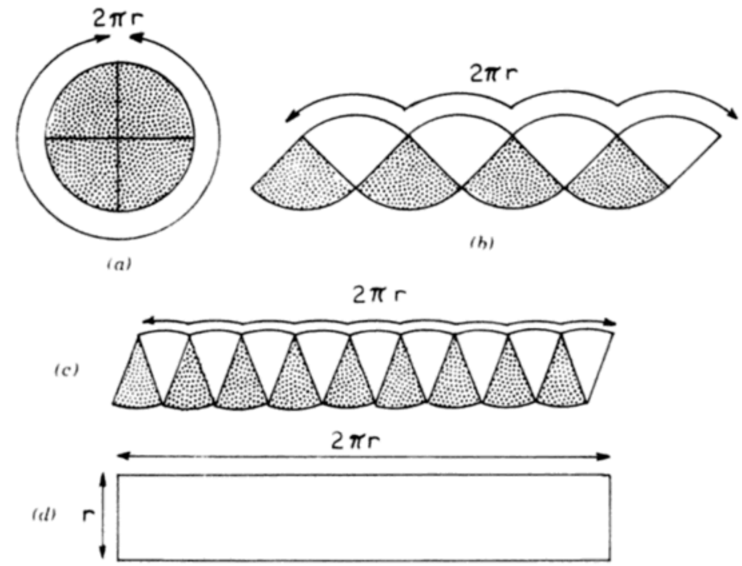
$$A = \pi r^2 \tag{5}$$

where r is the radius of the circle. Most of us first learned this formula in school with the justification that teacher said so, take it or leave it, but you better take it and learn it by heart; the formula is, in fact, an example of the brutality with which mathematics is often taught to the innocent. Those who later take a course in the integral calculus learn that the derivation of (5) is quite easy (see figure above). But how did people calculate the area of a circle almost five millenia before the integral calculus was invented?

They probably did it by a method of rearrangement. They calculated the area of a rectangle as length times width. To calculate the area of a parallelogram, they could construct a rectangle of equal area by rearrangement as in the figure below, and thus they found that the area of a parallelogram is given by base times height. The age of rigor that came with the later Greeks was still far away; they



The parallelogram and the rectangle have equal areas, as seen by cutting off the shaded triangle and reinserting it as indicated.



Determination of the area of a circle by rearrangement. The areas of the figures (b), (c), (d) equal exactly double the area of circle (a).

did not have to know about congruent triangles to be convinced by the "obvious" validity of the rearrangement.

So now let us try to use the general idea of rearrangement as in the figure above to convert a circle to a parallelogram of equal area. We are still using sticks to draw pictures in the sand, but this time we do this only to help our imagination, not to perform an actual measurement.

We first cut up a circle into four quadrants as in (a) above, and arrange them as shown in figure (b). Then we fill in the spaces between the segments by four equally large quadrants. The outline of the resulting weird figure is vaguely reminiscent of a parallelogram. The length of the figure, measured along the circular arcs, is equal to the circumference of the original circle, $2\pi r$. What we can say with certainty is that the area of this figure is exactly double the area of the original circle.

If we now divide the circle not into four, but into very many segments, our quasi-parallelogram (c) will resemble a parallelogram

The rearrangement method used in a 17th century Japanese document.[4]

much more closely; and the area of the circle is still exactlly one half of the quasi-parallelogram (c).

On continuing this process by cutting up the original circle into a larger and larger number of segments, the side formed by the little arcs of the segments will become indistinguishable from a straight line, and the quasi-parallelogram will turn into a true parallelogram (a rectangle) with sides $2\pi r$ and r. Hence the area of the circle is half of this rectangle, or πr^2.

The same construction can be seen in the Japanese document above (1698). Leonardo da Vinci also used this method in the 16th century. He did not have much of a mathematical education, and in any case, he could use little else, for Europe in his day, debilitated by more than a millenium of Roman Empire and Roman Church, was on a mathematical level close to that achieved in ancient Mesopotamia. It seems probable, then, that this was the way in which ancient peoples found the area of the circle.

And that should be our last speculation. From now on, we can rely on recorded history.